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Daubechies and Legendre Polynomials

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Abstract: We show the relationship between the Daubechies polynomials and the modified Legendre polynomials.

Key words: Shifted Legendre polynomials - Daubechies polynomials

INTRODUCTION

The wavelets are very important in science, engineering and technology [1-4], in particular, the construction of Daubechies wavelets [5] depends strongly from the zeros of Daubechies polynomials $d_l(x)$ [6, 7], thus it is interesting to study the properties of these polynomials because their behavior gives useful information on the corresponding wavelets. Here we show that the analysis of $d_l(x)$ may be guided through the modified Legendre polynomials $P_l^*(x)$ [8-18], therefore a better understanding of Daubechies polynomials can be obtained via the Legendre polynomials.

The shifted Legendre polynomials [13], for $x \, \varepsilon \, [0,1]$:

$$P_0^* = 1, \qquad P_1^* = 1 - 2x, \qquad P_2^* = 1 - 6x + 6x^2, \qquad P_3^* = 1 - 12x + 30x^2 - 20x^3,$$
$$P_4^* = 1 - 20x + 90x^2 - 140x^3 + 70x^4, \quad P_5^* = 1 - 30x + 210x^2 - 560x^3 + 630x^4 - 252x^5, \ldots \tag{1}$$

Are solutions of the differential equation:

$$x(1-x)\,y'' - (2x-1)\,y' + l(l+1)\,y = 0, \tag{2}$$

And they can be generated via the expression:

$$P_l^*(x) = \sum_{k=0}^{l} (-1)^k \binom{l}{k}\binom{l+k}{k} x^k, \qquad l = 0, 1, 2, 3, \ldots \tag{3}$$

Or in terms of the Gauss hypergeometric function [11, 12, 19, 20]:

$$P_l^*(x) = {}_2F_1(-l, l+1; 1; x). \tag{4}$$

We can indicate similarities between the shifted Legendre and Daubechies polynomials, in fact, in (2) we realize a simple change into the coefficient of y':

$$x(1-x)\,y'' - (2x+l)\,y' + l(l+1)\,y = 0, \tag{5}$$

Then it is nice to discover that the Daubechies polynomials [6, 7]:

$$d_0 = 1, \qquad d_1 = 1 + 2x, \qquad d_2 = 1 + 3x + 6x^2, \qquad d_3 = 1 + 4x + 10x^2 + 20x^3, \tag{6}$$

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$d_4 = 1 + 5x + 15x^2 + 35x^3 + 70x^4, \qquad d_5 = 1 + 6x + 21x^2 + 56x^3 + 126x^4 + 252x^5, \ldots$
Are solutions of (5).

Fig. 1 shows the polynomials (6), where only we can see their real roots; in general, their zeros are complex, for example, the roots of d_6 are (0.1411 + 0.3421 i), (-0.1246 + 0.2832 i), (-0.2665 + 0.1073 i) and their conjugates. Besides, there we note that $d_l(0) = 1$ and $d_l(x) > 0$ if $x > 0$, thus the real zeros are negative.

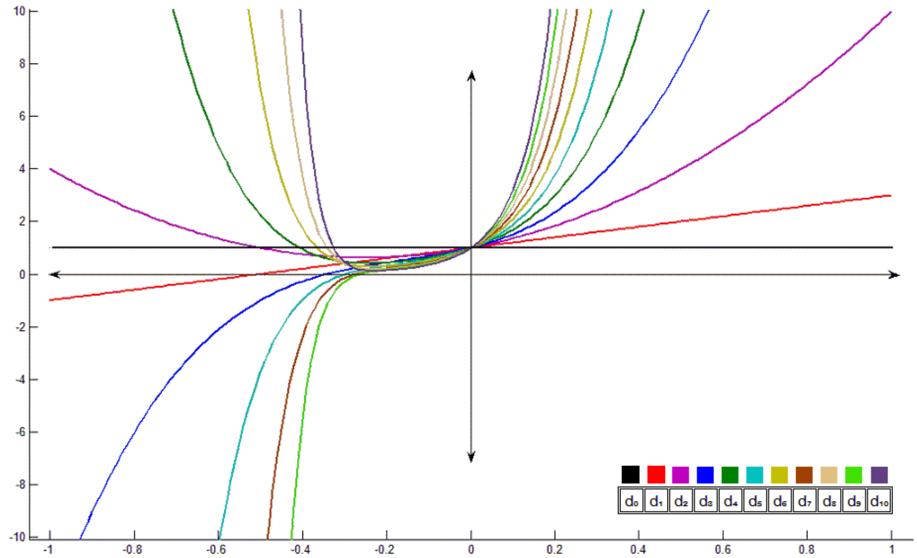

Fig. 1: Daubechies polynomials

The d_l are very important in the construction of the compactly supported Daubechies wavelets. There is a close relationship between the zeros of d_l and the $2l$ filter coefficients $h(l)$ of the Daubechies wavelets D_{2l} [6], therefore, it is fundamental to search efficient algorithms to find the roots of Daubechies polynomials, especially for large *l*. Here we show certain connections between (1) and (6), and then we hope that the stored experience with the roots of Legendre polynomials may be useful in the analysis of the zeros of (6).
It is easy to find the corresponding modification of (4):

$$d_l(x) = lim_{\lambda\to 0}\ {}_2F_1(-l, l+1; -l+\lambda; x)\,, \quad 0 < \lambda < 1, \tag{7}$$

Hence (3) adopts the known form [6, 7]:

$$d_l(x) = \sum_{k=0}^{l} \binom{l+k}{k} x^k. \tag{8}$$

Thus, in the equation:

$$x(1-x)\,y'' - (2x+\alpha)\,y' + l(l+1)\,y = 0, \tag{9}$$

We have two cases of interest:

$$y(x) = \begin{cases} P_l^*, & \alpha = -1, \\ d_l\,, & \alpha = l, \end{cases} \tag{10}$$

With the Rodrigues formulae:

$$P_l^*(x) = \frac{1}{l!}\frac{d^l}{dx^l}[x(1-x)]^l\,, \qquad d_l(x) = \frac{1}{l!}\left(\frac{x}{x-1}\right)^{l+1}\frac{d^l}{dx^l}\left[\frac{(x-1)^{2l+1}}{x}\right], \tag{11}$$

Which generate to (1) and (6). The expression:

$$y(x) = \frac{1}{(l+1)!}\,[b + l + (1-b)(-1)^l]\,\left(\frac{x}{x-1}\right)^{b+l}\,\frac{d^l}{dx^l}\left[\frac{(x-1)^{b+2l}}{x^b}\right], \tag{12}$$

Reproduces the relations (11) for $b = -l$ and $b = 1$, respectively.

From (8) we obtain the relations:

$$d_l(x) = \sum_{k=0}^{l} d_{lk}\, x^k\,, \qquad d_{lk} = \binom{l+k}{k}\,, \qquad k = 0, 1, \dots, l \tag{13}$$

Then it is immediate to deduce that:

$$d_{l0} = 1, \qquad d_{ll} = 2\, d_{l,l-1}\,, \qquad d_{lj} = \sum_{k=0}^{j} d_{l-1,k}\,, \quad j = 1, \dots, l-1, \tag{14}$$

This means that the coefficients of $d_{l-1}(x)$ allow to construct the next $d_l(x)$; we note the following property of Daubechies polynomials:

$$(1-x)^{l+1}\, d_l(x) + x^{l+1}\, d_l(1-x) = 1. \tag{15}$$

The shifted Legendre polynomials verify the three-term recurrence relation [18, 21]:

$$(l+1)\, P^*_{l+1}(x) = (2l+1)(1-2x)\, P^*_l(x) - l\, P^*_{l-1}(x), \tag{16}$$

Which implies their orthogonality:

$$\int_0^1 P^*_l(x)\; P^*_{l'}(x)\, dx = \frac{1}{2l+1}\, \delta_{ll'}, \tag{17}$$

However, the $d_l(x)$ are not orthogonal polynomials because they don’t satisfy a three-term recurrence expression.

REFERENCES

1. Hernández, E. and G. Weiss, 1996. A first course on wavelets, CRC Press, London.
2. Nievergelt, Y., 1999. Wavelets made easy, Birkhäuser, Berlin.
3. Jerri, A.J., 2007. Wavelets, Sampling Pub., New York.
4. Ryan, O., 2019. Linear algebra, signal processing and wavelets – a unified approach, Springer, Switzerland.
5. Daubechies, I., 1992. Ten lectures on wavelets, SIAM, Philadelphia.
6. Temme, N.M., 1997. Asymptotics and numerics of zeros of polynomials that are related to Daubechies wavelets, Appl. Comp. Harmonic Analysis, 4: 414-428.
7. Scipioni, A., P. Rischette and J.P. Préaux, 2012. Pascal’s triangle: An origin of Daubechies polynomials and an analytic expression for associated filter coefficients, Signal Processing, 92(1): 276-280.
8. Sommerfeld, A., 1964. Partial differential equations in physics, Academic Press, New York.
9. Lanczos, C., Legendre versus Chebyshev polynomials, in ‘Topics in Numerical Analysis’, (Proc. Roy. Irish Acad. Conf. on Numerical Analysis, Aug. 14-18, 1972), Ed. J. J. H. Miller, Academic Press, London, pp: 191-201.
10. Chihara, T.S., 1978. An introduction to orthogonal polynomials, Gordon & Breach, New York.
11. Hochstadt, H., 1986. The functions of Mathematical Physics, Dover, New York.
12. Oldham, K.B. and J. Spanier, 1987. An atlas of functions, Hemisphere Pub. Co., London.
13. Lanczos, C., 1988. Applied analysis, Dover, New York.
14. López-Bonilla, J., R. López-Vázquez and H. Torres-Silva, 2015. On the Legendre polynomials, Prespacetime Journal, 6(8): 735-739.
15. Doman, B.G.S., 2016. The classical orthogonal polynomials, World Scientific, Singapore.
16. Álvarez-Ballesteros, S., J. López-Bonilla and R. López-Vázquez, 2017. On the roots of the Legendre, Laguerre, and Hermite polynomials, Borneo Science J., 38(2): 41-45.
17. Gómez-Barrera, B., J. López-Bonilla and A. Ortega-Balcazar, 2019. Some properties of Legendre polynomials, Studies in Nonlinear Sci., 4(3): 32-33.
18. Foupouagnigni, M. and W. Koepf [Eds.], 2020. Orthogonal polynomials, Springer, Switzerland.

19. Dutka, J., 1984. The early history of the hypergeometric function, Arch. Hist. Exact Sci., 31(1): 15-34.
20. Seaborn, J.B., 1991. Hypergeometric functions and their applications, Springer-Verlag, New York.
21. Koepf, W. and D. Schmersau, 2002. Recurrence equations and their classical orthogonal polynomial solutions, Appl. Math. Comput., 128: 303-327.