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Dirac Spinor

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Abstract: For a given Lorentz matrix, we deduce the Dirac spinor's transformation in terms of four complex quantities.

Key words: Dirac spinor - Lorentz transformation

INTRODUCTION

The arbitrary complex quantities $\alpha, \beta, \gamma, \delta$ verifying the condition $\alpha\delta - \beta\gamma = 1$, generate a Lorentz matrix $L = (L^{\mu}{}_{\nu})$ via the expressions [1-9]:

$$L^0{}_0 = \tfrac{1}{2}\left(\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma} + \delta\bar{\delta}\right), \quad L^1{}_0 = \tfrac{1}{2}\left(\bar{\alpha}\,\gamma + \bar{\beta}\delta\right) + cc, \quad L^2{}_0 = -\tfrac{i}{2}\left(\alpha\bar{\gamma} - \bar{\beta}\delta\right) + cc,$$

$$L^0{}_1 = \tfrac{1}{2}(\bar{\alpha}\beta + \bar{\gamma}\delta) + cc, \quad L^1{}_1 = \tfrac{1}{2}(\bar{\alpha}\delta + \beta\bar{\gamma}) + cc, \quad L^2{}_1 = -\tfrac{i}{2}\left(\alpha\bar{\delta} + \beta\bar{\gamma}\right) + cc,$$

$$L^0{}_2 = -\tfrac{i}{2}(\bar{\alpha}\beta + \bar{\gamma}\delta) + cc, \quad L^1{}_2 = -\tfrac{i}{2}(\bar{\alpha}\delta + \beta\bar{\gamma}) + cc, \quad L^2{}_2 = \tfrac{1}{2}\left(\bar{\alpha}\delta - \bar{\beta}\gamma\right) + cc, \tag{1}$$

$$L^0{}_3 = \tfrac{1}{2}\left(\alpha\bar{\alpha} - \beta\bar{\beta} + \gamma\bar{\gamma} - \delta\bar{\delta}\right), \quad L^1{}_3 = \tfrac{1}{2}\left(\bar{\alpha}\gamma - \bar{\beta}\delta\right) + cc, \quad L^2{}_3 = -\tfrac{i}{2}\left(\alpha\bar{\gamma} + \bar{\beta}\delta\right) + cc,$$

$$L^3{}_0 = \tfrac{1}{2}\left(\alpha\bar{\alpha} + \beta\bar{\beta} - \gamma\bar{\gamma} - \delta\bar{\delta}\right), \quad L^3{}_1 = \tfrac{1}{2}(\bar{\alpha}\beta - \bar{\gamma}\delta) + cc, \quad L^3{}_2 = -\tfrac{i}{2}(\bar{\alpha}\beta - \bar{\gamma}\delta) + cc,$$

$$L^3{}_3 = \tfrac{1}{2}\left(\alpha\bar{\alpha} - \beta\bar{\beta} - \gamma\bar{\gamma} + \delta\bar{\delta}\right), \qquad \alpha\delta - \beta\gamma = 1,$$

where *cc* means the complex conjugate of all the previous terms.

The inverse problem is to obtain $\alpha, \beta, \gamma, \delta$ if we know L, and the answer is [9-12]:

$$\alpha = \frac{1}{D}\, Q^1{}_1 = \frac{1}{2D}[L^0{}_0 + L^0{}_3 + L^1{}_1 + L^2{}_2 + L^3{}_0 + L^3{}_3 - i\,(L^1{}_2 - L^2{}_1)],$$

$$\beta = \frac{1}{D}\, Q^1{}_2 = \frac{1}{2D}[L^0{}_1 + L^1{}_0 - L^1{}_3 + L^3{}_1 + i\,(L^0{}_2 + L^2{}_0 - L^2{}_3 + L^3{}_2)],$$

$$\gamma = \frac{1}{D}\, Q^2{}_1 = \frac{1}{2D}[L^0{}_1 + L^1{}_0 + L^1{}_3 - L^3{}_1 - i\,(L^0{}_2 + L^2{}_0 + L^2{}_3 - L^3{}_2)], \tag{2}$$

$$\delta = \frac{1}{D}\, Q^2{}_2 = \frac{1}{2D}[L^0{}_0 - L^0{}_3 + L^1{}_1 + L^2{}_2 - L^3{}_0 + L^3{}_3 + i\,(L^1{}_2 - L^2{}_1)],$$

where $D^2 = Q^1{}_1\, Q^2{}_2 - Q^1{}_2\, Q^2{}_1$.

On the other hand, the Dirac spinor obeys the transformation law [13, 14]:

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$$\tilde{\psi} = S\,\psi\,. \tag{3}$$

For a non-singular matrix S such that:

$$L^{\mu}{}_{\nu}\, S\, \gamma^{\nu} = \gamma^{\mu}\, S\,, \tag{4}$$

and we must determine a solution of (4) for a given Lorentz transformation. We have the expansion [15]:

$$\begin{gathered} S = b_0\, I + i d_0\, \gamma^5 + b_1\, \sigma^{23} + b_2\, \sigma^{31} + b_3\, \sigma^{12} + \sum_{j=1}^{3} d_j\, \sigma^{oj}, \\ b_0^2 - d_0^2 + \sum_{j=1}^{3} (d_j^2 - b_j^2) = 1, \qquad b_0\, d_0 - \sum_{j=1}^{3} b_j\, d_j = 0, \end{gathered} \tag{5}$$

In terms of Dirac matrices in the standard representation [13].

From (4) are immediate the expressions [14, 16]:

$$L^{\mu}{}_{0} = \frac{1}{4}\, tr\, (\gamma^0\, S^{-1} \gamma^{\mu}\, S), \qquad L^{\mu}{}_{k} = -\frac{1}{4}\, tr\, (\gamma^k\, S^{-1}\, \gamma^{\mu}\, S), \quad \mu = 0, \dots, 3, \quad k = 1, 2, 3, \tag{6}$$

that is, if we know *S* then with (6) we can determine the Lorentz matrix; (6) generates the relations:

$$\begin{aligned}
&L^0{}_0 = 2(b_0^2 - b_1^2 - b_2^2 - b_3^2) - 1, && L^0{}_1 = 2[\, (b_2\, d_3 - b_3\, d_2) + i\, (b_0\, d_1 - b_1\, d_0)\,], \\
&L^0{}_2 = 2[\, (b_3\, d_1 - b_1\, d_3) + i\, (b_0\, d_2 - b_2\, d_0)\,], && L^0{}_3 = 2[\, (b_1\, d_2 - b_2\, d_1) + i\, (b_0\, d_3 - b_3\, d_0)\,], \\
&L^1{}_0 = 2[-\, (b_2\, d_3 - b_3\, d_2) + i\, (b_0\, d_1 - b_1\, d_0)\,], && L^1{}_1 = 2[\, (b_0^2 - b_1^2) + (d_2^2 + d_3^2)\,] - 1, \\
&L^1{}_2 = 2[-(b_1\, b_2 + d_1\, d_2) - i\, (b_0\, b_3 + d_0\, d_3)\,], && L^1{}_3 = 2[-\, (b_1\, b_3 + d_1\, d_3) + i\, (b_0\, b_2 + d_0\, d_2)\,], \\
&L^2{}_0 = 2[-(b_3\, d_1 - b_1\, d_3) + i\, (b_0\, d_2 - b_2\, d_0)\,], && L^2{}_1 = 2[-\, (b_1\, b_2 + d_1\, d_2) + i\, (b_0\, b_3 + d_0\, d_3)\,], \\
&L^2{}_2 = 2[\, (b_0^2 - b_2^2) + (d_1^2 + d_3^2)\,] - 1, && L^2{}_3 = 2[-(b_2\, b_3 + d_2\, d_3) - i\, (b_0\, b_1 + d_0\, d_1)\,], \\
&L^3{}_0 = 2[-(b_1\, d_2 - b_2\, d_1) + i\, (b_0\, d_3 - b_3\, d_0)\,], && L^3{}_1 = 2[-\, (b_1\, b_3 + d_1\, d_3) - i\, (b_0\, b_2 + d_0\, d_2)\,], \\
&L^3{}_2 = 2[-(b_2\, b_3 + d_2\, d_3) + i\, (b_0\, b_1 + d_0\, d_1)\,], && L^3{}_3 = 2[\, (b_0^2 - b_3^2) + (d_1^2 + d_2^2)\,] - 1,
\end{aligned} \tag{7}$$

Which allow to obtain L if we have the expansion (5). However, here we have the inverse problem, that is, to obtain b_μ & d_μ, $\mu = 0, \dots, 3$ verifying (7) for a given Lorentz matrix. Our answer is the following:

$$\begin{gathered} b_0 = \frac{1}{4}\big(\alpha + \bar{\alpha} + \delta + \bar{\delta}\big),\ b_1 = \frac{1}{4}\big(\bar{\beta} - \beta + \bar{\gamma} - \gamma\big),\ b_2 = \frac{i}{4}\,\big(\beta + \bar{\beta} - \gamma - \bar{\gamma}\big),\ b_3 = \frac{1}{4}\big(\bar{\alpha} - \alpha + \delta - \bar{\delta}\big), \\ d_0 = \frac{i}{4}\big(\alpha - \bar{\alpha} + \delta - \bar{\delta}\big),\ d_1 = -\frac{i}{4}\big(\bar{\beta} + \beta + \bar{\gamma} + \gamma\big),\ d_2 = \frac{1}{4}\,\big(\bar{\beta} - \beta + \gamma - \bar{\gamma}\big),\ d_3 = \frac{i}{4}\big(\bar{\delta} + \delta - \alpha - \bar{\alpha}\big), \end{gathered} \tag{8}$$

Hence the expressions (1) are deduced if we apply (8) into (7). Besides, with (8) the matrix (5) acquires the structure:

$$S = \begin{pmatrix} A & E \\ E & A \end{pmatrix}, \qquad A = \frac{1}{2}\begin{pmatrix} \bar{\alpha} + \delta & \bar{\beta} - \gamma \\ \bar{\gamma} - \beta & \alpha + \bar{\delta} \end{pmatrix}, \qquad E = \frac{1}{2}\begin{pmatrix} \bar{\alpha} - \delta & \bar{\beta} + \gamma \\ \bar{\gamma} + \beta & \bar{\delta} - \alpha \end{pmatrix}. \tag{9}$$

Therefore, for a given Lorentz transformation first we employ (2) to determine $\alpha, \beta, \gamma, \delta$, then S is immediate via (9); this approach is an alternative to the process showed in [15] and to the explicit general formula obtained by Macfarlane [16]:

$$S = \frac{1}{4\sqrt{G}}\, [G\, I + \frac{i}{2}\, \varepsilon_{\mu\nu\alpha\beta}\, L^{\mu\nu}\, L^{\alpha\beta}\, \gamma^5 + i\, \Gamma(L^2) - i\, (2 + tr\, L)\, \Gamma(L)], \tag{10}$$

Such that:

$$G = 2\,(1 + tr\,L) + \frac{1}{2}[\,(tr\,L)^2 - tr\,L^2\,], \qquad tr\,L = \sum_{\mu=0}^{3} L^{\mu}{}_{\mu}\,, \qquad tr\,L^2 = \sum_{\nu,\alpha=0}^{3} L^{\nu}{}_{\alpha}\,L^{\alpha}{}_{\nu}\,,$$

$$\Gamma(L) = \sum_{\mu,\nu=0}^{3} L_{\mu\nu}\,\sigma^{\mu\nu}, \qquad \Gamma(L^2) = \sum_{\alpha,\mu,\nu=0}^{3} L_{\mu\alpha}\,L^{\alpha}{}_{\nu}\,\sigma^{\mu\nu}. \qquad (11)$$

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