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Lorentz Mapping and Dirac Spinor

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Abstract: We show a method to obtain the transformation of the Dirac spinor under Lorentz transformations, in particular, for boosts and 3-rotations.

Key words: Lorentz mapping - Dirac and Pauli matrices - Rotations and Boosts - Dirac 4-spinor

INTRODUCTION

We have the Dirac equation for spin-1/2 particles [1-5] $[(x^\mu) = (t, x, y, z), \hbar = c = 1]$:

$$\left(i\gamma^\mu \partial_\mu - m_0\right)\psi = 0, \qquad i = \sqrt{-1}, \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}, \tag{1}$$

where ψ is a 4-spinor with the γ^μ matrices verifying the anticommutator [6-8]:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\, I_{4x4}\,, \qquad (g^{\mu\nu}) = Diag(1, -1, -1, -1). \tag{2}$$

Here we shall use the Dirac-Pauli (or standard) representation [2, 9]:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \; j = 1, 2, 3, \tag{3}$$

With the Cayley [10]-Sylvester [11]-Pauli [12] matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

To analyze the transformation law of ψ under the orthochronic and proper Lorentz group [13-19]:

$$\tilde{x}^\mu = L^\mu{}_\nu\, x^\nu\,, \tag{5}$$

Which implies the existence [2, 7, 20, 21] of a non-singular matrix S such that:

$$L^\mu{}_\nu\, S\, \gamma^\nu = \gamma^\mu\, S\,, \tag{6}$$

and we deduce the relativistic invariance of (1) if the Dirac 4-spinor obeys the transformation rule:

$$\tilde{\psi} = S\,\psi\,. \tag{7}$$

In this work we use the representation (3) to study the matrix S, especially for boosts and 3-rotations. In fact, S can be expanded in terms of the sixteen Dirac matrices, in the standard representation [2, 3, 5]:

$$I, \qquad \gamma^\mu\,, \qquad \gamma^5\,, \qquad \gamma^\mu\,\gamma^5\,, \qquad \sigma^{\mu\nu}\,, \tag{8}$$

That is [8]:

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$$S = a_0\,\gamma^0\gamma^5 + b_0 I + c_0\,\gamma^0 + i d_0\,\gamma^5 + b_1\sigma^{23} + b_2\sigma^{31} + b_3\sigma^{12} + \sum_{j=1}^{3}(\,a_j\gamma^j + c_j\,\gamma^j\gamma^5 + d_j\,\sigma^{oj}), \quad (9)$$

and we must determine a solution of (6) for a given Lorentz transformation. The properties [2, 22]:

$$\det S = 1, \quad \gamma^0\,S^\dagger\,\gamma^0\,S = I, \quad [S, \gamma^5] = O, \quad S^\dagger\begin{pmatrix} O & I \\ -I & O \end{pmatrix} S = \begin{pmatrix} O & I \\ -I & O \end{pmatrix}, \quad (10)$$

Imply the relations:

$$b_0^* = b_0, \quad d_0^* = d_0, \quad b_j^* = -b_j, \quad d_j^* = -d_j, \;\; j = 1,2,3\,, \quad a_\mu = c_\mu = 0, \;\; \mu = 0, \dots, 3\,, \quad (11)$$

Thus (9) acquires the following structure:

$$S = b_0\,I + i d_0\,\gamma^5 + b_1\,\sigma^{23} + b_2\,\sigma^{31} + b_3\,\sigma^{12} + \sum_{j=1}^{3} d_j\,\sigma^{oj}, \quad (12)$$

SAuch that:

$$b_0^2 - d_0^2 + \sum_{j=1}^{3}(\,d_j^2 - b_j^2) = 1, \qquad b_0\,d_0 - \sum_{j=1}^{3} b_j\,d_j = 0. \quad (13)$$

From (6) are immediate the expressions [3, 22]:

$$L^\mu{}_0 = \frac{1}{4}\,tr\,(\gamma^0\,S^{-1}\,\gamma^\mu\,S), \qquad L^\mu{}_k = -\frac{1}{4}\,tr\,(\gamma^k\,S^{-1}\,\gamma^\mu\,S), \quad \mu = 0, \dots, 3, \quad k = 1,2,3, \quad (14)$$

That is, if we know S then with (14) we can determine the Lorentz matrix; (14) generates the relations:

$$\begin{aligned}
&L^0{}_0 = 2(b_0^2 - b_1^2 - b_2^2 - b_3^2) - 1, && L^0{}_1 = 2[\,(b_2\,d_3 - b_3\,d_2) + i\,(b_0\,d_1 - b_1\,d_0)\,],\\
&L^0{}_2 = 2[\,(b_3\,d_1 - b_1\,d_3) + i\,(b_0\,d_2 - b_2\,d_0)\,], && L^0{}_3 = 2[\,(b_1\,d_2 - b_2\,d_1) + i\,(b_0\,d_3 - b_3\,d_0)\,],\\
&L^1{}_0 = 2[-(b_2\,d_3 - b_3\,d_2) + i\,(b_0\,d_1 - b_1\,d_0)\,], && L^1{}_1 = 2[\,(b_0^2 - b_1^2) + (d_2^2 + d_3^2)\,] - 1,\\
&L^1{}_2 = 2[-(b_1\,b_2 + d_1\,d_2) - i\,(b_0\,b_3 + d_0\,d_3)\,], && L^1{}_3 = 2[-(b_1\,b_3 + d_1\,d_3) + i\,(b_0\,b_2 + d_0\,d_2)\,],\\
&L^2{}_0 = 2[-(b_3\,d_1 - b_1\,d_3) + i\,(b_0\,d_2 - b_2\,d_0)\,], && L^2{}_1 = 2[-(b_1\,b_2 + d_1\,d_2) + i\,(b_0\,b_3 + d_0\,d_3)\,],\\
&L^2{}_2 = 2[\,(b_0^2 - b_2^2) + (d_1^2 + d_3^2)\,] - 1, && L^2{}_3 = 2[-(b_2\,b_3 + d_2\,d_3) - i\,(b_0\,b_1 + d_0\,d_1)\,],\\
&L^3{}_0 = 2[-(b_1\,d_2 - b_2\,d_1) + i\,(b_0\,d_3 - b_3\,d_0)\,], && L^3{}_1 = 2[-(b_1\,b_3 + d_1\,d_3) - i\,(b_0\,b_2 + d_0\,d_2)\,],\\
&L^3{}_2 = 2[-(b_2\,b_3 + d_2\,d_3) + i\,(b_0\,b_1 + d_0\,d_1)\,], && L^3{}_3 = 2[\,(b_0^2 - b_3^2) + (d_1^2 + d_2^2)\,] - 1,
\end{aligned} \quad (15)$$

Which allow to obtain L if we have the expansion (12). Besides, we must remember the following identities satisfied by any Lorentz matrix:

$$\begin{aligned}
&(L^0{}_0)^2 = 1 + \sum_{j=1}^{3}(L^j{}_0)^2 = 1 + \sum_{j=1}^{3}(L^0{}_j)^2, \qquad L^0{}_\mu\,L^0{}_\nu = \sum_{j=1}^{3} L^j{}_\mu\,L^j{}_\nu, \quad \mu,\nu = 0, \dots, 3, \;\; \mu \neq \nu\\
&(L^0{}_r)^2 = \sum_{j=1}^{3}(L^j{}_r)^2 - 1, \qquad (L^r{}_0)^2 = \sum_{j=1}^{3}(L^r{}_j)^2 - 1, \qquad r = 1,2,3.
\end{aligned} \quad (16)$$

Now we shall consider the inverse problem, that is, to construct S for a given Lorentz transformation, then the fundamental formula (6) gives interesting information:

$$(1 - L^0{}_0)\,\vec{b} = M_0\,\vec{d}, \qquad (1 + L^0{}_0)\,\vec{d} = M_0\,\vec{b}, \qquad \vec{b} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}, \quad \vec{d} = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{pmatrix}, \quad (17)$$

$$M_0=\begin{pmatrix} 0 & -iL^0{}_1 & -iL^0{}_2 & -iL^0{}_3 \\ -iL^0{}_1 & 0 & L^0{}_3 & -L^0{}_2 \\ -iL^0{}_2 & -L^0{}_3 & 0 & L^0{}_1 \\ -iL^0{}_3 & L^0{}_2 & -L^0{}_1 & 0 \end{pmatrix}, \quad M_0^2=[1-(L^0{}_0)^2\,]\,I, \quad M_0^\dagger=-M_0,$$

$$L^j{}_0\,\vec{b}=M_j\,\vec{d}, \qquad L^j{}_0\,\vec{d}=N_j\,\vec{b}, \qquad M_j^\dagger=-M_j, \quad N_j^\dagger=-N_j, \qquad j=1,2,3, \tag{18}$$

$$M_jN_j=N_jM_j=(L^j{}_0)^2\,I, \qquad det\,M_j=det\,N_j=(L^j{}_0)^2,$$

such that:

$$M_1=\begin{pmatrix} 0 & i(1+L^1{}_1) & iL^1{}_2 & iL^1{}_3 \\ i(1+L^1{}_1) & 0 & -L^1{}_3 & L^1{}_2 \\ iL^1{}_2 & L^1{}_3 & 0 & (1-L^1{}_1) \\ iL^1{}_3 & -L^1{}_2 & -(1-L^1{}_1) & 0 \end{pmatrix}, \; N_1=\begin{pmatrix} 0 & i(1-L^1{}_1) & -iL^1{}_2 & -iL^1{}_3 \\ i(1-L^1{}_1) & 0 & L^1{}_3 & -L^1{}_2 \\ -iL^1{}_2 & -L^1{}_3 & 0 & (1+L^1{}_1) \\ -iL^1{}_3 & L^1{}_2 & -(1+L^1{}_1) & 0 \end{pmatrix},$$

$$M_2=\begin{pmatrix} 0 & iL^2{}_1 & i(1+L^2{}_2) & iL^2{}_3 \\ iL^2{}_1 & 0 & -L^2{}_3 & -(1-L^2{}_2) \\ i(1+L^2{}_2) & L^2{}_3 & 0 & -L^2{}_1 \\ iL^2{}_3 & (1-L^2{}_2) & L^2{}_1 & 0 \end{pmatrix},$$

$$N_2=\begin{pmatrix} 0 & -iL^2{}_1 & i(1-L^2{}_2) & -iL^2{}_3 \\ -iL^2{}_1 & 0 & L^2{}_3 & -(1+L^2{}_2) \\ i(1-L^2{}_2) & -L^2{}_3 & 0 & L^2{}_1 \\ -iL^2{}_3 & (1+L^2{}_2) & -L^2{}_1 & 0 \end{pmatrix}, \tag{19}$$

$$M_3=\begin{pmatrix} 0 & iL^3{}_1 & iL^3{}_2 & i(1+L^3{}_3) \\ iL^3{}_1 & 0 & (1-L^3{}_3) & L^3{}_2 \\ iL^3{}_2 & -(1-L^3{}_3) & 0 & -L^3{}_1 \\ i(1+L^3{}_3) & -L^3{}_2 & L^3{}_1 & 0 \end{pmatrix},$$

$$N_3=\begin{pmatrix} 0 & -iL^3{}_1 & -iL^3{}_2 & i(1-L^3{}_3) \\ -iL^3{}_1 & 0 & (1+L^3{}_3) & -L^3{}_2 \\ -iL^3{}_2 & -(1+L^3{}_3) & 0 & L^3{}_1 \\ i(1-L^3{}_3) & L^3{}_2 & -L^3{}_1 & 0 \end{pmatrix}.$$

Now we shall realize two applications of our expressions (16), … , (19):

- Boost in the *x* direction.

In this case the Lorentz matrix has the structure [13, 23]:

$$L=\begin{pmatrix} cosh\,\varphi & -\,sinh\,\varphi & 0 & 0 \\ -\,sinh\,\varphi & sinh\,\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad sinh\,\varphi=\frac{v}{\sqrt{1-v^2}}\,, \qquad cosh\varphi=\frac{1}{\sqrt{1-v^2}}\,, \tag{20}$$

Then (17), … , (20) imply $b_\mu=d_\mu=0$ except $d_1=i\,tanh\left(\frac{\varphi}{2}\right)\cdot b_0$, thus from (12) and (13):

$$b_0=cosh\left(\frac{\varphi}{2}\right),\;\; d_1=i\,sinh\left(\frac{\varphi}{2}\right),\;\; S=cosh\left(\frac{\varphi}{2}\right)I+i\,sinh\left(\frac{\varphi}{2}\right)\sigma^{01}=\begin{pmatrix} cosh\,(\frac{\varphi}{2})\cdot I & -sinh\,(\frac{\varphi}{2})\cdot\sigma_1 \\ -sinh\,(\frac{\varphi}{2})\cdot\sigma_1 & cosh(\frac{\varphi}{2})\cdot I \end{pmatrix},$$

In agreement with the literature [23-28].

- Rotation around the *x* axis.

Now the Lorentz matrix has the form [29, 30]:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}, \tag{21}$$

Hence from (16), … , (19) and (21) we deduce that $d_\mu = 0,\ \mu = 0, \dots, 3$ & $b_2 = b_3 = 0$, besides $b_1 = -i\,\tan\left(\frac{\theta}{2}\right) \cdot b_0$, with the relations (12) and (13), therefore $b_0 = \cos\left(\frac{\theta}{2}\right),\ b_1 = -i\,\sin\left(\frac{\theta}{2}\right)$:

$$S = \cos\left(\frac{\theta}{2}\right) I - i\,\sin\left(\frac{\theta}{2}\right) \sigma^{23} = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) I - i\,\sin\left(\frac{\theta}{2}\right) \cdot \sigma_1 & O \\ O & \cos\left(\frac{\theta}{2}\right) I - i\,\sin\left(\frac{\theta}{2}\right) \cdot \sigma_1 \end{pmatrix},$$

In accordance with Straub [23-28], but we note that he uses active rotations. We emphasize that our results are valid in the Dirac-Pauli scheme (3).

We consider that it is simple our method based in the expressions (17), (18) and (19) for a given Lorentz transformation, but it is interesting to indicate the explicit general formula obtained by Macfarlane [22]:

$$S = \frac{1}{4\sqrt{G}} \left[G\, I + \frac{i}{2}\, \varepsilon_{\mu\nu\alpha\beta}\, L^{\mu\nu}\, L^{\alpha\beta}\, \gamma^5 + i\, \Gamma(L^2) - i\,(2 + tr\, L)\, \Gamma(L) \right], \tag{22}$$

Such that:

$$G = 2\,(1 + tr\, L) + \frac{1}{2}\left[(tr\, L)^2 - tr\, L^2 \right], \quad tr\, L = \sum_{\mu=0}^{3} L^{\mu}{}_{\mu}\,, \quad tr\, L^2 = \sum_{\nu,\alpha=0}^{3} L^{\nu}{}_{\alpha}\, L^{\alpha}{}_{\nu}\,,$$
$$\Gamma(L) = \sum_{\mu,\nu=0}^{3} L_{\mu\nu}\, \sigma^{\mu\nu}, \qquad \Gamma(L^2) = \sum_{\alpha,\mu,\nu=0}^{3} L_{\mu\alpha}\, L^{\alpha}{}_{\nu}\, \sigma^{\mu\nu}. \tag{23}$$

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