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# Lorentz Transformation and its Associated Unimodular Matrix

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**Abstract:** We employ the factorization of the Lorentz matrix to construct the corresponding unimodular matrix.

**Key words:** Unimodular matrix, Lorentz mapping, Factorization of the Lorentz matrix, Special relativity.

## INTRODUCTION

The Lorentz matrix $L = (L^{\mu}{}_{\nu})$ between the frames of reference $(x^{\nu}) = (ct, x, y, z)$ and $(\tilde{x}^{\mu})$ has six degrees of freedom:

$$\tilde{x}^{\mu} = L^{\mu}{}_{\nu}\, x^{\nu}\,, \tag{1}$$

and it accepts the factorization [1-4]:

$$L = L_1\, L_2\,, \qquad L_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \gamma & \delta & \alpha & \beta \\ i\gamma & i\delta & -i\alpha & -i\beta \\ \alpha & \beta & -\gamma & -\delta \end{pmatrix}, \qquad L_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} \bar{\alpha} & \bar{\beta} & i\bar{\beta} & \bar{\alpha} \\ \bar{\beta} & \bar{\alpha} & -i\bar{\alpha} & -\bar{\beta} \\ \bar{\gamma} & \bar{\delta} & i\bar{\delta} & \bar{\gamma} \\ \bar{\delta} & \bar{\gamma} & -i\bar{\gamma} & -\bar{\delta} \end{pmatrix}, \tag{2}$$

Such that $det\, L_1 = -\, det\, L_2 = i$ and $\alpha$, $\beta$, $\gamma$, $\delta$ are arbitrary complex numbers [Cayley-Klein parameters [5]] verifying the condition $\alpha\,\delta - \beta\,\gamma = 1$. The corresponding inverse matrices are given by:

$$L_1^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} \delta & -\beta & i\beta & \delta \\ -\gamma & \alpha & -i\alpha & -\gamma \\ -\beta & \delta & i\delta & \beta \\ \alpha & -\gamma & -i\gamma & -\alpha \end{pmatrix}, \qquad L_2^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} \bar{\delta} & -\bar{\gamma} & -\bar{\beta} & \bar{\alpha} \\ -\bar{\gamma} & \bar{\delta} & \bar{\alpha} & -\bar{\beta} \\ i\bar{\gamma} & i\bar{\delta} & -i\bar{\alpha} & -i\bar{\beta} \\ \bar{\delta} & \bar{\gamma} & -\bar{\beta} & -\bar{\alpha} \end{pmatrix}, \tag{3}$$

Therefore:

$$L^{-1} = L_2^{-1}\; L_1^{-1} = \begin{pmatrix} L^0{}_0 & -L^1{}_0 & -L^2{}_0 & -L^3{}_0 \\ -L^0{}_1 & L^1{}_1 & L^2{}_1 & L^2{}_1 \\ -L^0{}_2 & L^1{}_2 & L^2{}_2 & L^3{}_2 \\ -L^0{}_3 & L^1{}_3 & L^2{}_3 & L^3{}_3 \end{pmatrix}. \tag{4}$$

From (2) are immediate the elements of the Lorentz matrix [4, 6-12]:

$$\begin{aligned}
&L^0{}_0 = \tfrac{1}{2}\left(\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma} + \delta\bar{\delta}\right), \quad && L^1{}_0 = \tfrac{1}{2}\left(\bar{\alpha}\,\gamma + \bar{\beta}\delta\right) + cc, \quad && L^2{}_0 = -\,\tfrac{i}{2}\left(\alpha\bar{\gamma} - \bar{\beta}\delta\right) + cc\,, \\
&L^0{}_1 = \tfrac{1}{2}(\bar{\alpha}\beta + \bar{\gamma}\delta) + cc, && L^1{}_1 = \tfrac{1}{2}(\bar{\alpha}\delta + \beta\bar{\gamma}) + cc, && L^2{}_1 = -\,\tfrac{i}{2}\left(\alpha\bar{\delta} + \beta\bar{\gamma}\right) + cc, \\
&L^0{}_2 = -\,\tfrac{i}{2}(\bar{\alpha}\beta + \bar{\gamma}\delta) + cc, && L^1{}_2 = -\,\tfrac{i}{2}(\bar{\alpha}\delta + \beta\bar{\gamma}) + cc, && L^2{}_2 = \tfrac{1}{2}\left(\bar{\alpha}\delta - \bar{\beta}\gamma\right) + cc\,, \\
&L^0{}_3 = \tfrac{1}{2}\left(\alpha\bar{\alpha} - \beta\bar{\beta} + \gamma\bar{\gamma} - \delta\bar{\delta}\right), && L^1{}_3 = \tfrac{1}{2}\left(\bar{\alpha}\gamma - \bar{\beta}\delta\right) + cc, && L^2{}_3 = -\,\tfrac{i}{2}\left(\alpha\bar{\gamma} + \bar{\beta}\delta\right) + cc\,,
\end{aligned} \tag{5}$$

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$L^3{}_0 = \frac{1}{2}\left(\alpha\bar{\alpha} + \beta\bar{\beta} - \gamma\bar{\gamma} - \delta\bar{\delta}\right), \qquad L^3{}_1 = \frac{1}{2}(\bar{\alpha}\beta - \bar{\gamma}\delta) + cc, \qquad L^3{}_2 = -\frac{i}{2}(\bar{\alpha}\beta - \bar{\gamma}\delta) + cc,$

$L^3{}_3 = \frac{1}{2}\left(\alpha\bar{\alpha} - \beta\bar{\beta} - \gamma\bar{\gamma} + \delta\bar{\delta}\right), \qquad \alpha\delta - \beta\gamma = 1,$

where *cc* means the complex conjugate of all the previous terms. The expression (4) is in agreement with the known result [8]:

$$L^{-1\,a}{}_b = \eta_{(b)(c)}\, L^c{}_r\, \eta^{(r)(a)}\,. \tag{6}$$

The unimodular matrix $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ generates to $L$, therefore $L^{-1}$ is generated by $B^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$, then if in (5) we make the changes $\alpha \to \delta,\ \beta \to -\beta,\ \gamma \to -\gamma,\ \delta \to \alpha$ we obtain the properties:

$$L^{-1\,0}{}_0 = L^0{}_0\,, \qquad L^{-1\,0}{}_j = -L^j{}_0\,, \qquad L^{-1\,j}{}_0 = -L^0{}_j\,, \qquad L^{-1\,j}{}_k = L^k{}_j\,, \tag{7}$$

which is equivalent to (4).

If we know the matrix $B$, then with (5) we construct the corresponding Lorentz matrix; the inverse problem is to obtain $B$ if we have $L$, and the answer is [4, 13-16]:

$$B = (B^A{}_C) = \pm\frac{1}{K}\, L^\mu{}_\nu\, \sigma_\mu{}^{A\dot{E}}\, \sigma^\nu{}_{C\dot{E}}, \qquad K \equiv \sqrt{\det\left(L^\tau{}_\lambda\, \sigma_\tau{}^{Q\dot{F}} \sigma^\lambda{}_{R\dot{F}}\right)}\,, \tag{8}$$

In terms of the Infeld-van der Waerden symbols [9, 17-19]. Here we shall employ the factorization (2) to exhibit an alternative procedure to (8), in fact, we have the relation $L_1^{-1}\, L = L_2$ where we can apply (2) and (3) to obtain the expressions:

$$\begin{aligned}
\bar{\alpha} &= \delta\,(L^0{}_0 + L^3{}_0) + \beta\,(-L^1{}_0 + i\,L^2{}_0) = \delta\,(L^0{}_3 + L^3{}_3) + \beta\,(-L^1{}_3 + i\,L^2{}_3),\\
&= -\gamma\,(L^0{}_1 + L^3{}_1) + \alpha\,(L^1{}_1 - i\,L^2{}_1) = -i\,\gamma\,(L^0{}_2 + L^3{}_2) + \alpha\,(L^2{}_2 - i\,L^1{}_2),\\
\bar{\beta} &= -\gamma\,(L^0{}_0 + L^3{}_0) + \alpha\,(L^1{}_0 - i\,L^2{}_0) = \gamma\,(L^0{}_3 + L^3{}_3) - \alpha\,(L^1{}_3 - i\,L^2{}_3),\\
&= \delta\,(L^0{}_1 + L^3{}_1) + \beta\,(-L^1{}_1 + i\,L^2{}_1) = -i\,\delta\,(L^0{}_2 + L^3{}_2) + \beta\,(L^2{}_2 + i\,L^2{}_2),\\
\bar{\gamma} &= \beta\,(-L^0{}_0 + L^3{}_0) + \delta\,(L^1{}_0 + i\,L^2{}_0) = \beta\,(-L^0{}_3 + L^3{}_3) + \delta\,(L^1{}_3 + i\,L^2{}_3),\\
&= \alpha\,(L^0{}_1 - L^3{}_1) - \gamma\,(L^1{}_1 + i\,L^2{}_1) = i\,\alpha\,(L^0{}_2 - L^3{}_2) - \gamma\,(-L^2{}_2 + i\,L^1{}_2),\\
\bar{\delta} &= \alpha\,(L^0{}_0 - L^3{}_0) - \gamma\,(L^1{}_0 + i\,L^2{}_0) = \alpha\,(L^3{}_3 - L^0{}_3) + \gamma\,(L^1{}_3 + i\,L^2{}_3),\\
&= \beta\,(L^3{}_1 - L^0{}_1) + \delta\,(L^1{}_1 + i\,L^2{}_1) = i\,\beta\,(L^0{}_2 - L^3{}_2) + \delta\,(L^2{}_2 - i\,L^1{}_2).
\end{aligned} \tag{9}$$

Two applications of our relations (9):

a). Boost [8, 20, 21].

$$L = \begin{pmatrix} \cosh\varphi & -\sinh\varphi & 0 & 0 \\ -\sinh\varphi & \sinh\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \sinh\varphi = \frac{v}{\sqrt{1-v^2}}\,, \qquad \cosh\varphi = \frac{1}{\sqrt{1-v^2}}\,, \tag{10}$$

Then from (9) it is immediate the following information:

$$\bar{\beta} = \beta = \gamma = -\,\delta\tanh\left(\frac{\varphi}{2}\right), \qquad \bar{\alpha} = \alpha = \delta, \tag{11}$$

Thus the constraint $\alpha\delta - \beta\gamma = 1$ implies the known values [22]:

$$\alpha = \delta = cosh\left(\frac{\varphi}{2}\right), \qquad \gamma = \beta = -\,sinh\left(\frac{\varphi}{2}\right). \tag{12}$$

b). Rotation in the plane YZ [2].

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & cos\,\theta & -\,sin\,\theta \\ 0 & 0 & sin\,\theta & cos\,\theta \end{pmatrix}, \tag{13}$$

In this case the conditions (9) give the properties:

$$\bar{\beta} = -\beta = -\gamma = i\,\delta\,tan\left(\frac{\theta}{2}\right), \qquad \bar{\alpha} = \alpha = \delta, \tag{14}$$

but $det\,B = 1$, hence [22]:

$$\delta = \alpha = cos\left(\frac{\theta}{2}\right), \qquad \beta = \gamma = i\,sin\left(\frac{\theta}{2}\right). \tag{15}$$

Thus we observe that it is easy the use of (9) to determine the unimodular matrix corresponding to a given Lorentz transformation.

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