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Dirac Spinor Under 3-Rotations and Boosts

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Abstract: For boosts and 3-rotations, we determine the matrix of transformation of the Dirac spinor.

Key words: Dirac matrices - Lorentz mappings - Pauli matrices - Dirac 4-spinor - Boosts and Rotations

INTRODUCTION

We have the Dirac equation for spin-1/2 particles [1-9] $[(x^\mu) = (t, x, y, z), \hbar = c = 1]$:

$$(i\gamma^\mu \partial_\mu - m_0)\psi = 0, \qquad i = \sqrt{-1}, \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}, \tag{1}$$

where ψ is a 4-spinor with the γ^μ matrices verifying the anticommutator [10-11]:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\, I_{4x4}\,, \qquad (g^{\mu\nu}) = Diag(1, -1, -1, -1). \tag{2}$$

Here we shall use the Dirac-Pauli (or standard) representation [3, 12]:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix}, \; j = 1, 2, 3, \tag{3}$$

With the Cayley [13]-Sylvester [14]-Pauli [15] matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

To study the transformation law of ψ under the homogeneous Lorentz group [16-23]:

$$\tilde{x}^\mu = L^\mu{}_\nu\, x^\nu\,, \tag{5}$$

Which implies the existence [3, 24, 25] of a non-singular matrix S such that:

$$L^\mu{}_\nu\, S\, \gamma^\nu = \gamma^\mu\, S\,, \tag{6}$$

and we obtain the relativistic invariance of (1) if the Dirac spinor obeys the transformation rule:

$$\tilde{\psi} = S\,\psi\,. \tag{7}$$

In this work we determine S for boosts and 3-rotations, using the representation (3). In fact, S can be expanded in terms of the sixteen Dirac matrices, in the standard representation [3, 7, 9]:

$$I, \qquad \gamma^\mu\,, \qquad \gamma^5\,, \qquad \gamma^\mu\,\gamma^5\,, \qquad \sigma^{\mu\nu}\,, \tag{8}$$

That is [26]:

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$$S = a_0\,\gamma^0\gamma^5 + b_0 I + c_0\,\gamma^0 + id_0\,\gamma^5 + b_1\sigma^{23} + b_2\sigma^{31} + b_3\sigma^{12} + \sum_{j=1}^{3}(a_j\gamma^j + c_j\,\gamma^j\gamma^5 + d_j\,\sigma^{oj}), \qquad (9)$$

and we must obtain a solution of (6) for a given Lorentz transformation. Hence (9) implies the relations:

$$\begin{aligned}
\gamma^0 S &= id_0\,\gamma^0\gamma^5 + c_0\,I + b_0\,\gamma^0 + a_0\,\gamma^5 + c_1\,\sigma^{23} + c_2\,\sigma^{31} + c_3\,\sigma^{12} + id_1\,\gamma^1 + id_2\,\gamma^2 + id_3\,\gamma^3 + \\
&\quad + b_1\,\gamma^1\gamma^5 + b_2\,\gamma^2\gamma^5 + b_3\,\gamma^3\gamma^5 - ia_1\,\sigma^{01} - ia_2\,\sigma^{02} - ia_3\,\sigma^{03}, \\
\gamma^1 S &= b_1\,\gamma^0\gamma^5 - a_1\,I + id_1\,\gamma^0 - c_1\,\gamma^5 - a_0\,\sigma^{23} + ia_3\,\sigma^{31} - ia_2\,\sigma^{12} + b_0\,\gamma^1 - ib_3\,\gamma^2 + ib_2\,\gamma^3 + \\
&\quad + id_0\,\gamma^1\gamma^5 + d_3\,\gamma^2\gamma^5 - d_2\,\gamma^3\gamma^5 + ic_0\,\sigma^{01} + c_3\,\sigma^{02} - c_2\,\sigma^{03}, \\
\gamma^2 S &= b_2\,\gamma^0\gamma^5 - a_2\,I + id_2\,\gamma^0 - c_2\,\gamma^5 - ia_3\,\sigma^{23} - a_0\,\sigma^{31} + ia_1\,\sigma^{12} + ib_3\,\gamma^1 + b_0\,\gamma^2 - ib_1\,\gamma^3 - \\
&\quad - d_3\,\gamma^1\gamma^5 + id_0\,\gamma^2\gamma^5 + d_1\,\gamma^3\gamma^5 - c_3\,\sigma^{01} + ic_0\,\sigma^{02} + c_1\,\sigma^{03}, \\
\gamma^3 S &= b_3\,\gamma^0\gamma^5 - a_3\,I + id_3\,\gamma^0 - c_3\,\gamma^5 + ia_2\,\sigma^{23} - ia_1\,\sigma^{31} - a_0\,\sigma^{12} - ib_2\,\gamma^1 + ib_1\,\gamma^2 + b_0\,\gamma^3 + \\
&\quad + d_2\,\gamma^1\gamma^5 - d_1\,\gamma^2\gamma^5 + id_0\,\gamma^3\gamma^5 + c_2\,\sigma^{01} - c_1\,\sigma^{02} + ic_0\,\sigma^{03}, \\
L^\mu{}_\nu\, S\,\gamma^\nu &= (-id_0L^\mu{}_0 + b_1L^\mu{}_1 + b_2L^\mu{}_2 + b_3L^\mu{}_3)\,\gamma^0\gamma^5 + (c_0L^\mu{}_0 - a_1L^\mu{}_1 - a_2L^\mu{}_2 - a_3L^\mu{}_3)\,I + \\
&\quad + (b_0L^\mu{}_0 - id_1L^\mu{}_1 - id_2L^\mu{}_2 - id_3L^\mu{}_3)\,\gamma^0 + (-a_0L^\mu{}_0 + c_1L^\mu{}_1 + c_2L^\mu{}_2 + c_3L^\mu{}_3)\,\gamma^5 + \\
&\quad + (c_1L^\mu{}_0 - a_0L^\mu{}_1 + ia_3L^\mu{}_2 - ia_2L^\mu{}_3)\,\sigma^{23} + (c_2L^\mu{}_0 - ia_3L^\mu{}_1 - a_0L^\mu{}_2 + ia_1L^\mu{}_3)\,\sigma^{31} + \\
&\quad + (c_3L^\mu{}_0 + ia_2L^\mu{}_1 - ia_1L^\mu{}_2 - a_0L^\mu{}_3)\,\sigma^{12} + (-id_1L^\mu{}_0 + b_0L^\mu{}_1 - ib_3L^\mu{}_2 + ib_2L^\mu{}_3)\,\gamma^1 + \\
&\quad + (-id_2L^\mu{}_0 + ib_3L^\mu{}_1 + b_0L^\mu{}_2 - ib_1L^\mu{}_3)\,\gamma^2 + (-id_3L^\mu{}_0 - ib_2L^\mu{}_1 + ib_1L^\mu{}_2 + b_0L^\mu{}_3)\,\gamma^3 + \\
&\quad + (b_1L^\mu{}_0 - id_0L^\mu{}_1 - d_3L^\mu{}_2 + d_2L^\mu{}_3)\,\gamma^1\gamma^5 + (b_2L^\mu{}_0 + d_3L^\mu{}_1 - id_0L^\mu{}_2 - d_1L^\mu{}_3)\,\gamma^2\gamma^5 + \\
&\quad + (b_3L^\mu{}_0 - d_2L^\mu{}_1 + d_1L^\mu{}_2 - id_0L^\mu{}_3)\,\gamma^3\gamma^5 + (ia_1L^\mu{}_0 - ic_0L^\mu{}_1 - c_3L^\mu{}_2 + c_2L^\mu{}_3)\,\sigma^{01} + \\
&\quad + (ia_2L^\mu{}_0 + c_3L^\mu{}_1 - ic_0L^\mu{}_2 - c_1L^\mu{}_3)\,\sigma^{02} + (ia_3L^\mu{}_0 - c_2L^\mu{}_1 + c_1L^\mu{}_2 - ic_0L^\mu{}_3)\,\sigma^{03}.
\end{aligned} \qquad (10)$$

I. Boost in the direction X.

In this case the Lorentz matrix has the structure [16, 27]:

$$L = \begin{pmatrix} \cosh\varphi & -\sinh\varphi & 0 & 0 \\ -\sinh\varphi & \sinh\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \sinh\varphi = \frac{v}{\sqrt{1-v^2}}\,, \qquad \cosh\varphi = \frac{1}{\sqrt{1-v^2}}\,, \qquad (11)$$

thus (6), (10) and (11) imply the following information for each value of the index μ:

$\mu = 0$:

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = -\tanh\left(\frac{\varphi}{2}\right)\cdot\begin{pmatrix} c_1 \\ c_0 \\ ic_3 \\ -ic_2 \end{pmatrix}, \qquad \begin{pmatrix} id_0 \\ id_1 \\ d_2 \\ d_3 \end{pmatrix} = -\tanh\left(\frac{\varphi}{2}\right)\cdot\begin{pmatrix} b_1 \\ b_0 \\ b_3 \\ -b_2 \end{pmatrix}, \qquad (12)$$

$\mu = 1$:

$$\begin{pmatrix} b_2 \\ b_3 \end{pmatrix} = \tanh\left(\frac{\varphi}{2}\right)\cdot\begin{pmatrix} d_3 \\ -d_2 \end{pmatrix}, \qquad \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = -\tanh\left(\frac{\varphi}{2}\right)\cdot\begin{pmatrix} a_1 \\ a_0 \end{pmatrix}, \qquad (13)$$

Then from (12) and (13) we have that $a_j = c_j = 0,\ j = 0,1$ and $b_k = d_k = 0,\ k = 2,3,$ hence (12) has a reduction:

$$\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = i\,tanh\left(\frac{\varphi}{2}\right) \cdot \begin{pmatrix} -c_3 \\ c_2 \end{pmatrix}, \qquad \begin{pmatrix} d_0 \\ d_1 \end{pmatrix} = i\,tanh\left(\frac{\varphi}{2}\right) \cdot \begin{pmatrix} b_1 \\ b_0 \end{pmatrix}, \tag{14}$$

$\mu = 2$: We obtain the values $a_3 = b_1 = c_2 = d_0 = 0$, and finally $\mu = 3$ gives $a_2 = c_3 = 0$, therefore (14) implies the relation:

$$d_1 = i\,tanh\left(\frac{\varphi}{2}\right) \cdot b_0, \tag{15}$$

and from (9) we deduce the expansion:

$$S = b_0\, I + d_1\, \sigma^{01} = b_0\left[I + i\,tanh\left(\frac{\varphi}{2}\right) \cdot \sigma^{01}\right] = b_0 \begin{pmatrix} I & -tanh\,(\frac{\varphi}{2}) \cdot \sigma_1 \\ -tanh\,(\frac{\varphi}{2}) \cdot \sigma_1 & I \end{pmatrix},$$

But $b_0 = cosh(\frac{\varphi}{2})$ to verify the condition $\det S = 1$ [3], then S for a boost in the direction X is given by:

$$S = \begin{pmatrix} cosh\,(\frac{\varphi}{2}) \cdot I & -sinh\,(\frac{\varphi}{2}) \cdot \sigma_1 \\ -sinh\,(\frac{\varphi}{2}) \cdot \sigma_1 & cosh(\frac{\varphi}{2}) \cdot I \end{pmatrix}, \tag{16}$$

In agreement with the literature [27-31].

II. Rotation in the plane YZ.

Now the Lorentz matrix has the form [32]:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & cos\,\theta & -\sin\theta \\ 0 & 0 & sin\,\theta & cos\,\theta \end{pmatrix}, \tag{17}$$

Then from (6), (10) and (17):

$$\begin{gathered} \mu = 0\text{: Implies } a_k = d_k = 0,\ k = 0,\dots,4\,; \qquad \mu = 1\text{: Gives } b_2 = b_3 = c_0 = c_1 = 0, \\ \mu = 2\text{: We obtain that } b_1 = -i\tan\left(\frac{\theta}{2}\right) \cdot b_0,\quad c_2 = \tan(\frac{\theta}{2}) \cdot c_3\,; \qquad \mu = 3\text{: Implies } c_2 = c_3 = 0, \end{gathered} \tag{18}$$

Hence from (9) and (18) the matrix of transformation for the Dirac spinor acquires the structure:

$$S = b_0\left[I - i\ tan\left(\frac{\theta}{2}\right) \cdot \sigma^{23}\right] = b_0 \begin{pmatrix} I - i\,tan\ (\frac{\theta}{2}) \cdot \sigma_1 & O \\ O & I - i\,tan\ (\frac{\theta}{2}) \cdot \sigma_1 \end{pmatrix},$$

But the condition $\det S = 1$ gives $b_0 = cos\ (\frac{\theta}{2})$, therefore:

$$S = \begin{pmatrix} cos\left(\frac{\theta}{2}\right) I - i\,sin\,(\frac{\theta}{2}) \cdot \sigma_1 & O \\ O & cos\left(\frac{\theta}{2}\right) I - i\,sin\,(\frac{\theta}{2}) \cdot \sigma_1 \end{pmatrix}, \tag{19}$$

In accordance with Straub [27-32], but we note that he uses active rotations. We emphasize that the results (16) and (19) are valid in the Dirac-Pauli scheme (3).

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