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On the Lah and Stirling Numbers

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Abstract: We employ the Lah numbers to give an elementary deduction of an identity obtained by Qi-Lim-Guo involving Stirling numbers

Key words: Stirling numbers - Qi-Lim-Guo's identity - Lah numbers

INTRODUCTION

The Lah numbers [1] are given by [2-4]:

$$(-1)^n L_n^{[j]} \equiv \sum_{k=0}^{n} (-1)^k S_n^{(k)} S_k^{[j]} = \frac{(-1)^n (n-1)!}{(j-1)!} \binom{n}{j}, \tag{1}$$

Involving the Stirling numbers [2, 5-7]. For example, if $j = 1, 2$, then from (1) we obtain the known relations:

$$\sum_{k=0}^{n} (-1)^k S_n^{(k)} = (-1)^n n!, \sum_{k=0}^{n} (-2)^k S_n^{(k)} = (-1)^n (n+1)!, \tag{2}$$

For Stirling numbers of the first kind.

Now we consider the following expression:

$$\sum_{k=0}^{n} (-1)^k S_n^{(k)} S_{k+1}^{[j+1]} = \sum_{k=0}^{n} (-1)^k S_n^{(k)} \left\{ S_k^{[j]} + (j+1) S_k^{[j+1]} \right\},$$

$$\overset{(1)}{=} (-1)^n L_n^{[j]} + (-1)^n (j+1) L_n^{[j+1]} = \frac{(-1)^n (n-1)!}{(j-1)!} \left[\binom{n}{j} + \frac{j+1}{j} \binom{n}{j+1} \right],$$

Therefore:

$$\sum_{k=0}^{n} (-1)^k S_n^{(k)} S_{k+1}^{[j+1]} = \frac{(-1)^n n!}{j!} \binom{n}{j}, \qquad 0 \le j \le n. \tag{3}$$

On the other hand, we have the inversion formula [8]:

$$\sum_{k=0}^{n} (-1)^{n-k} S_n^{(k)} f(k) = g(n) \quad \therefore \quad \sum_{k=0}^{n} (-1)^k S_n^{[k]} g(k) = f(n), \tag{4}$$

Whose application to (3) implies the identity:

$$\sum_{k=0}^{n} (-1)^k \binom{k}{j} k!\, S_n^{[k]} = (-1)^n j!\, S_{n+1}^{[j+1]}, \qquad 0 \le j \le n, \tag{5}$$

Obtained by Qi-Lim-Guo [9].

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We know the property [10]:

$$A \equiv (-1)^m \sum_{j=0}^{m} (-1)^j \binom{m}{j} \binom{z\,j}{n} = \frac{m!}{n!} \sum_{k=0}^{n} S_n^{(k)} S_k^{[m]} z^k , \quad n, m \geq 0, z \in \mathbb{C} , \qquad (6)$$

Deduced in [11-13]. We can give an elementary proof of (6), in fact, we have the relation [7]:

$$\binom{x}{n} = \frac{1}{n!} \sum_{r=0}^{n} S_n^{(r)} x^r \qquad \therefore \qquad \binom{z\,j}{n} = \frac{1}{n!} \sum_{r=0}^{n} S_n^{(r)} j^r z^r , \qquad (7)$$

Hence:

$$A = \frac{(-1)^m}{n!} \sum_{r=0}^{n} S_n^{(r)} z^r \left\{ \sum_{j=0}^{m} (-1)^j \binom{m}{j} j^r \right\} = \frac{(-1)^m}{n!} \sum_{r=0}^{n} S_n^{(r)} z^r \left\{ (-1)^m \, m! \; S_r^{[m]} \right\} = (6), \; q.e.d.$$

REFERENCES

1. Lah, I., 1954. A new kind of numbers and its application in the actuarial mathematics, Bol. do Inst. dos Actuários Portugueses, 9: 7-15.
2. Aigner, M., 2007. A course in enumeration, Springer, Berlin.
3. Daboul, S., J. Mangaldan, M Z. Spivey and P.J. Taylor, 2013. The Lah numbers and the nth derivative of exp(1/x), Math. Mag., 86(1): 39-47.
4. Boyadzhiev, K.N., 2016. Lah numbers, Laguerre polynomials of order negative one, and the nth derivative of exp(1/x), Acta Univ. Sapientiae-Mathematica, 8(1): 22-31.
5. Kauers, M. and P. Paule, 2011. The concrete tetrahedron, Springer, Berlin.
6. Boyadzhiev, K.N., 2012. Close encounters with the Stirling numbers of the second kind, Math. Mag., 85(4): 252-266.
7. Quaintance, J. and H.W. Gould, 2016. Combinatorial identities for Stirling numbers, World Scientific, Singapore.
8. Spivey, M.Z., 2019. The art of proving binomial identities, CRC Press, Boca Raton, Fl, USA.
9. Qi, F., D. Lim and Bai-Ni Guo, 2018. Some identities related to Eulerian polynomials and involving the Stirling numbers, Appl. Anal. Discrete Math., 12: 467-480. K.
10. Boyadzhiev, K.N., 2014. Power series with binomial sums and asymptotic expansions, Int. J. Math. Analysis, 8(28): 1389-1414.
11. Todorov, P.G., 1988. Taylor expansions of analytic functions related to $(1+z)^x - 1$, J. Math. Anal. Appl., 132: 264-280.
12. Ch. A. Charalambides and J. Singh, 1988. A review of the Stirling numbers, their generalizations and statistical interpretations, Comm. Statist. Theory Methods, 17(8): 2533-2595.
13. Ch. A. Charalambides, 2002. Enumerative combinatorics, Chapman and Hall / CRC, London.