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On a Conjecture Involving Stirling Numbers of The First Kind

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Abstract: We employ hypergeometric functions to study a conjecture involving Stirling numbers of the first kind.

Key words: Stirling numbers - Hypergeometric functions

INTRODUCTION

In [1] is the conjecture:

$$I(m,n) \equiv \sum_{k=\max(m,n)}^{\infty} (-1)^k \binom{k+1}{n+1} \frac{(n+1)^k}{k!} S_{k+1}^{(k+1-m)} = 0, \quad \forall\, m,n \geq 0, \tag{1}$$

Involving Stirling numbers of the first kind [2-4]. Here we use hypergeometric functions [5, 6] to verify (1) for $m = 0, 1, 2$ with n arbitrary. First we shall consider some special cases to show our approach:

a). $m = n = 0$.

$$I(0,0) = \sum_{k=0}^{\infty} t_k\,, \quad t_k = \frac{(-1)^k (k+1)}{k!} \quad \therefore \quad \frac{t_{k+1}}{t_k} = -\frac{k+2}{(k+1)^2},$$

Then [4, 7]:

$$I(0,0) = {}_1F_1(2; 1; -1) = 0, \tag{2}$$

where was applied the relation:

$${}_1F_1(a; a-1; z) = e^z \left(1 + \frac{z}{a-1}\right). \tag{3}$$

b). $m = 1, n = 0$.

$$I(1,0) = \sum_{k=1}^{\infty} \frac{(-1)^k (k+1)}{k!} S_{k+1}^{(k)} = 2 \sum_{j=0}^{\infty} t_j\,, \quad t_j = \frac{(-1)^j (j+2)^2}{4 \cdot j!} \quad \therefore \quad \frac{t_{j+1}}{t_j} = -\frac{(j+3)^2}{(j+2)^2 (j+1)},$$

$$= 2\ {}_2F_2(3,3; 2,2; -1) = 0, \tag{4}$$

where were employed the expressions:

$$S_{k+1}^{(k)} = -\frac{k\,(k+1)}{2}, \qquad {}_2F_2(a,a; a-1,a-1; z) = \frac{e^z}{(a-1)^2}\,[(z+a)^2 + 1 - z - 2\,a]. \tag{5}$$

c). $m = 2, n = 0$.

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$$I(2,0)=\sum_{k=2}^{\infty}\frac{(-1)^k(k+1)}{k!}\,S_{k+1}^{(k-1)}=3\sum_{j=0}^{\infty}t_j,\quad t_j=\frac{(-1)^j\,(j+3)^2(3j+8)}{72\cdot j!}\quad\therefore\quad\frac{t_{j+1}}{t_j}=-\frac{(j+4)^2(j+\frac{11}{3})}{(j+3)^2(j+\frac{8}{3})(j+1)},$$

$$\frac{1}{3}\,I(2,0)=\ {}_3F_3\left(4,4,\tfrac{11}{3};3,3,\tfrac{8}{3};\,-1\right)=\ {}_2F_2\left(4,\tfrac{11}{3};3,\tfrac{8}{3};\,-1\right)-\tfrac{11}{18}\ {}_2F_2\left(5,\tfrac{14}{3};4,\tfrac{11}{3};\,-1\right),\tag{6}$$

Because we used the formulae:

$$S_{k+1}^{(k-1)}=\frac{1}{24}\,(k+1)\,k\,(k-1)\,(3k+2),$$

$$\tag{7}$$

$${}_3F_3(a,a_2,a_3;a-1,b_2,b_3;z)=\ {}_2F_2(a_2,a_3;b_2,b_3;z)+\frac{a_2a_3\,z}{(a-1)b_2b_3}\ {}_2F_2(a_2+1,a_3+1;\ b_2+1,b_3+1;z).$$

We know the property:

$${}_2F_2(a,b;a-1,d;z)=\frac{b\,z}{(a-1)\,d}\ \ {}_1F_1(b+1;d+1;z)+\ {}_1F_1(b;d;z),\tag{8}$$

thus:

$${}_2F_2\left(4,\tfrac{11}{3};3,\tfrac{8}{3};\,-1\right)=-\frac{11}{24}\ {}_1F_1\left(\tfrac{14}{3};\,\tfrac{11}{3};-1\right)+\ {}_1F_1\left(\tfrac{11}{3};\,\tfrac{8}{3};-1\right)\overset{(3)}{=}\frac{7}{24}\,e^{-1},$$

$${}_2F_2\left(5,\tfrac{14}{3};4,\tfrac{11}{3};\,-1\right)=-\frac{7}{22}\ \ {}_1F_1\left(\tfrac{17}{3};\,\tfrac{14}{3};-1\right)+\ {}_1F_1\left(\tfrac{14}{3};\,\tfrac{11}{3};-1\right)\overset{(3)}{=}\frac{21}{44}\,e^{-1},$$

Then from (6):

$$I(2,0)=0.\tag{9}$$

d). $m=0,n=1.$

$$I(0,1)=\sum_{k=1}^{\infty}\frac{(-2)^k\,(k+1)}{(k-1)!}=-2\ \ {}_1F_1(3;2;-2)\overset{(3)}{=}0.\tag{10}$$

e). $m=n=1.$

$$I(1,1)=\sum_{k=1}^{\infty}\frac{(-2)^k\,(k+1)}{(k-1)!}S_{k+1}^{(k)}\overset{(5)}{=}2\ \ {}_2F_2(3,3;2,1;\,-2)\overset{(8)}{=}2[-3\ {}_1F_1(4;2;-2)+\ {}_1F_1(3;1;-2)],\tag{11}$$

But we have the relation:

$${}_1F_1(a;a-2;z)=\frac{2\,e^z}{(1-a)_2}\,L_2^{a-3}(-z),\tag{12}$$

Involving the associated Laguerre polynomials [5], therefore:

$${}_1F_1(4;2;-2)=-\frac{1}{3}\,e^{-2}\,,\qquad {}_1F_1(3;1;-2)=-e^{-2},$$

Hence from (11):

$$I(1,1)=0.\tag{13}$$

f). $m=2,n=1.$

$$I(2,1)=12\ {}_3F_3\left(4,4,\tfrac{11}{3};3,2,\tfrac{8}{3};-2\right)\overset{(7)}{=}12\left[\ {}_2F_2\left(4,\tfrac{11}{3};2,\tfrac{8}{3};-2\right)-\tfrac{11}{6}\ {}_2F_2\left(5,\tfrac{14}{3};3,\tfrac{11}{3}\ ;-2\right)\right]=0,\tag{14}$$

where we applied (7) and (8) to obtain:

$${}_2F_2\left(4,\tfrac{11}{3};2,\tfrac{8}{3};-2\right)=-\tfrac{3}{2}\ {}_1F_1(5;3;-2)+\ {}_1F_1(4;2;-2)=-\tfrac{1}{3}\ e^{-2},$$
$${}_2F_2\left(5,\tfrac{14}{3};3,\tfrac{11}{3};-2\right)=-\tfrac{10}{11}\ {}_1F_1(6;4;-2)+\ {}_1F_1(5;3;-2)=-\tfrac{2}{11}\ e^{-2}.$$

g). $m=0, n\geq 0$.

$$I(0,n)=\frac{(-1)^n\,(n+1)^n}{n!}\ {}_1F_1\big(n+2;\ n+1;-(n+1)\big)\overset{(3)}{=}0. \tag{15}$$

h). $m=1, n\geq 1$.

$$I(1,n)=-\frac{(-1)^n\,(n+1)^{n+1}}{2\cdot(n-1)!}\ {}_2F_2\big(n+2,n+2;\ n+1,n;-(n+1)\big),$$

$$\overset{(8)}{=}-\frac{(-1)^n\,(n+1)^{n+1}}{2\cdot(n-1)!}\left[-\frac{n+2}{n}\ {}_1F_1\big(n+3;\ n+1;-(n+1)\big)+\ {}_1F_1\big(n+2;\ n;-(n+1)\big)\right]=0, \tag{16}$$

Because (12) implies the values:

$${}_1F_1\big(n+3;\ n+1;-(n+1)\big)=-\frac{e^{-(n+1)}}{n+2},\qquad {}_1F_1\big(n+2;\ n;-(n+1)\big)=-\frac{e^{-(n+1)}}{n}.$$

i). $m=2, n\geq 2$.

$$I(2,n)=\frac{(-1)^n\,(n+1)^{n+1}\,(3n+2)}{24\cdot(n-2)!}\ A, \tag{17}$$

Such that:

$$A\equiv\ {}_3F_3\left(n+2,n+2,n+\tfrac{5}{3};\ n+1,n-1,n+\tfrac{2}{3};-(n+1)\right)=B-\frac{(n+2)\,(n+\frac{5}{3})}{(n-1)\,(n+\frac{2}{3})}\ C, \tag{18}$$

where:

$$B\equiv\ {}_2F_2\left(n+2,n+\tfrac{5}{3};n-1,n+\tfrac{2}{3};-(n+1)\right),$$

$$=-\frac{(n+1)\,(n+2)}{(n-1)\,(n+\frac{2}{3})}\ {}_1F_1\big(n+3;n;-(n+1)\big)+\ {}_1F_1\big(n+2;n-1;-(n+1)\big)=\frac{3n+\frac{7}{3}}{n(n-1)(n+\frac{2}{3})}\ e^{-(n+1)},$$

$$C\equiv\ {}_2F_2\left(n+3,n+\tfrac{8}{3};\ n,n+\tfrac{5}{3};\ -(n+1)\right),$$

$$=-\frac{(n+1)\,(n+3)}{n\,(n+\frac{5}{3})}\ {}_1F_1\big(n+4;n+1;-(n+1)\big)+\ {}_1F_1\big(n+3;n;-(n+1)\big)=\frac{3n+\frac{7}{3}}{n(n+2)(n+\frac{5}{3})}\ e^{-(n+1)},$$

Then (18) gives $A=0$ and from (17):

$$I(2,n)=0. \tag{19}$$

Therefore, the results (2, 4, 9, 10, 13-16, 19) imply:

$$I(m,n) = 0, \quad m = 0, 1, 2, \ \forall\, n \geq 0, \tag{20}$$

thus the validity of (1) for $m \geq 3$ and $n = 0, 1, 2, 3, \ldots$, is an open problem.

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