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Sun's Binomial Inversion Formula

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Abstract: We study some particular cases of the Sun's binomial inversion algorithm.

Key words: Stirling numbers - Binomial inversion - Bernoulli and Fubini numbers

INTRODUCTION

If we consider the binomial expression:

$$F(n) = \sum_{k=0}^{n} \binom{n}{k} f(k) - \lambda\, f(n), \quad n \geq 0, \tag{1}$$

Then Sun [1, 2] obtained the corresponding inversion formula:

$$f(n) = \frac{1}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} F(k)\, B_{n+1-k}\,, \quad \lambda = 1, \tag{2}$$

$$f(n) = -\sum_{m=0}^{n} \binom{n}{m} F(m) \sum_{k=0}^{n-m} \frac{k!}{(\lambda-1)^{k+1}}\, S_{n-m}^{[k]}\,, \quad \lambda \neq 1, \tag{3}$$

Involving Bernoulli and Stirling numbers [3-6].

We can give a simple deduction of the case (2), in fact, we know the relation [7]:

$$\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} B_{n-k} = \binom{n}{m}\left(B_{n-m} + \delta_{n-m,1}\right), \quad n \geq m \geq 0, \tag{4}$$

Thus:

$$\sum_{m=0}^{n} \sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} B_{n-k}\, f(m) = \sum_{m=0}^{n} \binom{n}{m} B_{n-m}\, f(m) + \binom{n}{n-1} f(n-1),$$

Therefore:

$$n\, f(n-1) = \sum_{k=0}^{n} \binom{n}{k} \left[\sum_{m=0}^{k} \binom{k}{m} f(m) - f(k)\right] B_{n-k}\,,$$

In agreement with (1) and (2) for $\lambda = 1$.

Now we shall study the case $\lambda = -1$, hence from (3):

$$f(n) = \sum_{m=0}^{n} \binom{n}{m} F(m) \sum_{k=0}^{n-m} \frac{(-1)^k\, k!}{2^{k+1}}\, S_{n-m}^{[k]}\,, \tag{5}$$

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But we have the identity [2-4, 8-11]:

$$\sum_{k=0}^{N} \frac{(-1)^k\, k!}{2^{k+1}}\, S_N^{[k]} = \frac{1-2^{N+1}}{N+1}\, B_{N+1}\,, \tag{6}$$

Then (5) implies the following inversion formula:

$$f(n) = \sum_{r=0}^{n} \binom{n}{r} \frac{1-2^{r+1}}{r+1}\, F(n-r)\, B_{r+1}\,, \tag{7}$$

Which is equivalent to (2) if $\lambda = -1$. For example, in [6] is the property:

$$2\,[n + (-1)^n B_n] = \sum_{k=0}^{n} \binom{n}{k} (-1)^k\, 2^k\, B_k + (-1)^n\, 2^n\, B_n\,, \tag{8}$$

With the structure (1), thus (7) implies the relation:

$$(-1)^n\, 2^{n-1}\, B_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1-2^{k+1}}{k+1}\, [n - k + (-1)^{n-k} B_{n-k}]\, B_{k+1}\,, \quad n \geq 0. \tag{9}$$

Now we shall consider the case $\lambda = 1$; for example, (8) can be written in the form:

$$2\,[n + (-1)^n(1-2^n)B_n] = \sum_{k=0}^{n} \binom{n}{k} (-1)^k\, 2^k\, B_k - (-1)^n\, 2^n\, B_n\,,$$

Then (2) allows deduce the identity:

$$(-1)^n\,(n+1)\, 2^{n-1}\, B_n = \sum_{k=0}^{n+1} \binom{n+1}{k} [k + (-1)^k(1-2^k)B_k]\, B_{n+1-k}\,, \quad n \geq 0. \tag{10}$$

The Namias expression [6, 12-16] has the form (1):

$$2\,(1-2^n)\, B_n = \sum_{k=0}^{n} \binom{n}{k} 2^k\, B_k - 2^n\, B_n\,,$$

Thus from (2):

$$n\, 2^{n-2}\, B_{n-1} = \sum_{k=0}^{n} \binom{n}{k} (1-2^k)\, B_k\, B_{n-k}\,, \quad n \geq 1. \tag{11}$$

In [4] we find the relation:

$$\frac{2(2^n-1)}{n+1} = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} - \frac{1}{n+1}, \tag{12}$$

Hence (2) implies the identity:

$$\sum_{k=0}^{n} \binom{n}{k} \frac{2^k - 1}{k+1}\, B_{n-k} = \frac{1}{2}\,, \quad n \geq 1. \tag{13}$$

Similarly [4]:

$$\frac{1-(-1)^n}{n+1} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k+1} - \frac{(-1)^n}{n+1}, \tag{14}$$

Then from (2):

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k - 1}{k+1}\, B_{n-k} = (-1)^n\,, \quad n \geq 1. \tag{15}$$

For the case $\lambda = 0$:

$$F(n) = \sum_{k=0}^{n} \binom{n}{k} f(k), \tag{16}$$

and (3) gives the well-known inversion [4, 17, 18]:

$$f(n) = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} F(m), \quad n \geq 0, \tag{17}$$

Where we applied the property [4]:

$$\sum_{k=0}^{N} (-1)^k \, k! \, S_N^{[k]} = (-1)^N. \tag{18}$$

If $\lambda = \frac{1}{2}$ then from (3):

$$f(n) = 2 \sum_{m=0}^{n} \binom{n}{m} F(m) \sum_{k=0}^{n-m} (-1)^k \, k! \, 2^k \, S_{n-m}^{[k]}, \tag{19}$$

But we have the Fubini numbers [11, 19-21]:

$$a(r) \equiv \sum_{k=1}^{r} k! \, S_r^{[k]} = \sum_{q=0}^{r} (-1)^{r-q} \, q! \, 2^q \, S_{r+1}^{[q+1]} = \frac{(-1)^r}{2} \sum_{k=0}^{r} (-1)^k \, k! \, 2^k \, S_r^{[k]}, \quad r \geq 1, \tag{20}$$

That is, $\{a(0), a(1), a(2), \ldots\} = \{1, 1, 3, 13, 75, 541, 4683, 47293, \ldots\}$, thus (19) implies the binomial inversion formula:

$$\frac{1}{2} f(n) = F(n) + 2 \sum_{m=0}^{n-1} (-1)^{n-m} \binom{n}{m} F(m) \, a(n-m), \quad n \geq 0. \tag{21}$$

Finally, for $\lambda = 2$ the expression (3) gives the relation:

$$f(n) = -\sum_{m=0}^{n} \binom{n}{m} F(m) \sum_{k=0}^{n-m} k! \, S_{n-m}^{[k]},$$

Where we can use (20) to deduce the inversion:

$$-f(n) = F(n) + \sum_{m=0}^{n-1} \binom{n}{m} F(m) \, a(n-m), \quad n \geq 0. \tag{22}$$

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