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Identities for Bernoulli and Stirling Numbers from the Roman's Formula

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Abstract: We use the Roman's combinatorial relation to obtain identities for the Bernoulli and Stirling numbers.

Key words: Stirling numbers ・ Roman's formula ・ Polynomials and numbers of Bernoulli

INTRODUCTION

Roman [1, 2] obtained the identity:

$$S_{n-1}^{[k-1]} = \frac{1}{n(k-1);}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k+j}B_n(j), \quad 1 \le k \le n, \tag{1}$$

where $B_n(x)$ and $S_r^{[m]}$ are the Bernoulli polynomials [3-5] and the Stirling numbers of the second kind [6-9], respectively:

$$B_n(j) = \sum_{r=0}^{n}\binom{n}{r}B_r j^{n-r}, \tag{2}$$

$$S_m^{[k]} = \frac{(-1)^k}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^m, \tag{3}$$

with $B_r \equiv B_r(0)$ the Bernoulli numbers [4-8] and is very known the relationship [10]:

$$B_r = \sum_{k=0}^{r}\frac{(-1)^k k!}{k+1}S_t^{[k]}, \tag{4}$$

and the convolution form [7]:

$$\sum_{r=0}^{n}\binom{n}{r}S_r^{[q]}\,S_{n-r}^{[k]} = \binom{q+k}{q}\,S_n^{[q+k]}. \tag{5}$$

In Sec. 2 we employ (1), … , (5) to deduce identities involving the Bernoulli and Stirling numbers.

Roman's Formula:
From (1) and (2):

$$S_{n-1}^{[k-1]} = \frac{(-1)^k}{n(k-1);}\sum_{r=0}^{n}\binom{n}{r}B_r\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^{n-r},$$

where we can use (3) to obtain the expression:

$$S_{n-1}^{[k-1]} = \frac{k}{n}\sum_{r=0}^{n}\binom{n}{r}B_r S_{n-r}^{[k]}. \tag{6}$$

The application of (4) into (6) gives:

$$S_{n-1}^{[k-1]} = \frac{k}{n}\sum_{q=0}^{n}\frac{(-1)^q q!}{q-1}\sum_{r=0}^{n}\binom{n}{r}S_r^{[q]}\,S_{n-r}^{[k]} \overset{(5)}{=} \frac{1}{n(k-1)!}\sum_{q=0}^{n-k}\frac{(-1)^q(q+k);}{q+1}S_n^{[q+k]},$$

therefore:

$$S_{n-1}^{[k-1]} = \frac{(-1)^k}{n(k-1)!}\sum_{j=k}^{n}\frac{(-1)^j j!}{j-k+1}S_n^{[j]}, \quad 1 \le k \le n. \tag{7}$$

Let's remember the recurrence relation [6, 7]:

$$S_n^{[k]} = S_{n-1}^{[k-1]} + k\,S_{n-1}^{[k]}, \tag{8}$$

where we can employ (7) to deduce the interesting identity:

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$$\sum_{r=0}^{n-k} \frac{(-1)^r (r+k)!(n-k+n\,r)}{r+1} S_{n+1}^{[r+k+1]} = 0, \quad 1 \le k \le n. \quad (9)$$

We know values of the Stirling numbers, for example, $S_n^{[0]} = \delta_{n0}$, $S_n^{[n]} = 1$, $S_m^{[1]} = 1$, $m \ge 1$, etc., then (7) implies the properties:

$$\sum_{k=0}^{n} (-1)^k \, k! \, S_{n+1}^{[k+1[} = \delta_{n0}, \quad \sum_{k=1}^{n-1} (-1)^k (k-1)! \, S_n^{[k+1]} = 1-n, \quad (10)$$

$$\sum_{k=0}^{n-1} (-1)^k (k+2)k! \, S_{n+1}^{[k+2]} = 1+n.$$

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