Computational and Applied Mathematical Sciences 4 (1): 07-09 2019
ISSN 2222-1328

DOI: 10.5829/idosi.cams.2019.07.09

Solving a System of Linear Equations by LU Decomposition and Cholesky Factorization

Awel Seid Geletie, Dereje Legesse Abaire and Ahmed Buseri Ashine

Department of Mathematics, Madda Walabu University, Bale Robe, Ethiopia

Abstract: In this paper, the solution of a system of linear equations using LU Decomposition and Cholesky Factorization using MATLAB is executed. Examples are given.

Key words: LU decomposition · Cholesky Factorization

INTRODUCTION

MATLAB is a highly resourceful and useful tool used by a majority of the academic community to create models, simulations and solve many problems from the fields of engineering, science and mathematics. It is particularly known for its ease in handling arrays and matrices. A lot of in-built functions are provided to make computation easy. In this paper, we have defined a MATLAB coding that is useful to solve a system of n linear equations with n variables. In the literature [1], we find that a matrix called pivot matrix is used to obtain the solution accordingly. In the pivot matrix multiplication, the main aim is to rearrange the columns elements of the coefficient matrix, so that the largest element of each column falls on the principal diagonal. This pivotization is more time taking. We suggest a new method to overcome these difficulties. In this paper, we have presented the brief definition and MATLAB coding of the Cholesky Factorization andLU Decomposition of the coefficient matrix of the linear equations and using this decomposition have found the solution of the system of equations.

Standard Notation and Terminology

Rank: The rank of a matrix A is the size of the largest collection of linearly independent columns of .

Coefficient Matrix: The coefficient matrix refers to a matrix consisting of the coefficients of the variables in a set of linear equation.

Augmented Matrix: An augmented matrix is a matrix obtained by appending the columns of coefficient matrix with the constant matrix, usually for the purpose of performing the same elementary row operations on each of the given matrices.

Inverse of a Matrix: For a square matrix A, the inverse is written A^{-1}. When A is multiplied by A^{-1} the result is the identity matrix I. Non-square matrices do not have inverses. Not all square matrices have inverses.

Zero Matrix: A zero matrix is the additive identity of the additive group of matrices.

Identity Matrix: A square matrix in which all the elements of the principal diagonal are ones and all other elements are zeros. The effect of multiplying a given matrix by an identity matrix is to leave the given matrix unchanged.

Positive Definite Matrix: AsymmetricnxnrealmatrixA is said to be positive definite if z^TAz is positive for every non-zero column vectorz of n real numbers. Here z^T denotes the transpose of z.

Matrix Multiplication: Let Am x p and Bp x n be two matrices whose elements are represented by, $A = [a_{ij}]$ & $B = [b_{ij}]$ for i = 1, 2, 3, … m and j = 1, 2, 3, ... n. Then their matrix product is given by [2], Cij = aik * bkj

System of n Linear Equations: A system of linear equations is a set or collection ofequations that are dealt with all together at once. A system of m equations in n variables is defined as,

Corresponding Author: Ahmed Buseri Ashine, Department of Mathematics, Madda Walabu University, Bale Robe, Ethiopia.

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$
$$.... \quad ... \quad ... \quad ... \quad .... \quad ...$$
$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m$$

where aij, bi, are real numbers and xj are variables for i = 1, 2, …, m, j = 1, 2, …, n. In matrix notation it is represented as, AX = b Three cases arise to the solution of this system of equations:

- Unique solution
- Infinite solutions
- No solution

The uniqueness of the solution ofthissystem can be seen using matrix algebra [3] by comparing the rank of the coefficient matrix A and augmented matrix K as follows (where n is the number of variables/unknowns).

- If rank(A) = rank(K) = n, there exists a unique solution.
- If rank(A) = rank(K) < n, there exist infinitely many solutions.

There are various methods to solve this system of equations like substitution, cross-multiplication, matrix algebra methods etc. We concentrate here on LU Decomposition method.

MATERIALS AND METHODS

Cholesky Factorization: In CholeskeyFactorization, every square matrix A can be written as A = U’*U where U is an upper triangular matrix and U’ is a transpose of U. This U is called Cholesky Factor.

Let A = $(a_{ij})_{nxn}$ be a square matrix. If $a_{ij} = a_{ji}$fro all i, j , the we say that A is a symmetric matrix.

Suppose, A is the coefficient matrix, B is the constant matrix, X is the variable matrix (whose elements are to be determined),

Given AX = B
U= chol(A), we can find U satisfying
U’*U=A
(U’*U)*X=B or U’*(U*X)=B
Therefore, U*X=U\B
Hence X=U\(U’\B)

LU Decomposition Method: In LU Decomposition [4], every square matrix A can be decomposed into a product of a lower triangular matrix L and an upper triangular matrix U. That is A = LU

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

The sufficient condition for a matrix A to be decomposable is positive definiteness. Now, for forming system of equations solvable for unique solution, one of the upper or lower triangular matrices should have 1’s in the diagonal elements.

Let AX = B
(LU)X = B since A =LU and L(UX) = B and UX = L\B
Hence X= U\(L\B)

Here, L is the lower triangular matrix (whose elements are to be determined) and U is the upper triangular matrix (whose elements are to be determined).

Examples: We verified the following examples in executed MATLAB coding by both methods.

Example 1: Consider the system of equations,

$$\begin{cases} 4x - 2y + z = 11 \\ -2x + 4y - 2z = -16 \\ x + -2y + 4z = 17 \end{cases}$$

Using LU decomposition:

```
>> A=[4 -2 1;-2 4 -2;1 -2 4];
>> B=[11;-16;17];
>> [L, U]=lu(A);
>> X=U\(L\B)
X =
1
-2
3
```

Using Cholesky Factorization:

```
>> A=[4 -2 1;-2 4 -2;1 -2 4];
>> B=[11;-16;17];
>> U=chol(A);
>> X=U\(U'\B)
X =
1.0000
-2.0000
3.0000
```

Example 1: Consider the system of equations:

$$\begin{cases} 2x_1 - x_2 + x_3 + 2x_4 + x_5 - 3x_6 = 20 \\ x_1 + x_2 - 2x_3 + x_4 + 3x_5 - x_6 = 4 \\ 4x_1 + 3x_2 + x_3 - 6x_4 - 3x_5 - 2x_6 = -27 \\ 5x_1 + 2x_2 - x_3 - x_4 + 2x_5 + x_6 = -3 \\ x_1 + 3x_2 - 3x_3 - x_4 + 2x_5 + x_6 = -15 \\ -3x_1 - x_2 + 2x_3 + 3x_4 + x_5 + 3x_6 = 16 \end{cases}$$

```
>> A=[1 1 -2 1 3 -1;2 -1 1 2 1 -3;1 3 -3 -1 2 1;5 2 -1 -1 2 1;-3
-1 2 3 1 3;4 3 1 -6 -3 -2]
A =
1 1 -2 1 3 -1
2 -1 1 2 1 -3
1 3 -3 -1 2 1
5 2 -1 -1 2 1
-3 -1 2 3 1 3
4 3 1 -6 -3 -2
>> B=[4;20;-15;-3;16;-27]
B =
4
20
-15
-3
16
-27

>> A=[1 1 -2 1 3 -1;2 -1 1 2 1 -3;1 3 -3 -1 2 1;5 2 -1 -1 2 1;-3
-1 2 3 1 3;4 3 1 -6 -3 -2];
>> B=[4;20;-15;-3;16;-27];
>> [L, U]=lu(A);
>> X=U\(L\B)
X =
1.0000
-2.0000
3.0000
4.0000
2.0000
-1.0000
```

REFERENCES

1. http://rosettacode.org/wiki/LU_decomposition - Creating a MATLAB function
2. Erwin Kreyszig, Advanced Engineering Mathematics, 9th Edition, pp: 279.
3. Erwin Kreyszig, Advanced Engineering Mathematics, 9th Edition, pp: 302.
4. Erwin Kreyszig, Advanced Engineering Mathematics, 9th Edition, pp: 841.
5. RudraPratap, Getting Started with MATLAB, Oxford University Press, 2010.