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# On an Identity for Stirling Numbers

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**Abstract:** We show that certain identity for Stirling numbers of the second kind coincides with an identity obtained by Roman and we give an alternative proof for it.

**Key words:** Stirling numbers

## INTRODUCTION

In the Web we find the following identity [1]:

$$Q = \sum_{j=k}^{n} (-1)^{n-j} \binom{n}{j-1} S_j^{[k]} = k\ S_n^{[k]}, \qquad (1)$$

involving the Stirling numbers of the second kind [2, 3]. Here we wish to point out that (1) is precisely the identity obtained by Roman [4]:

$$\sum_{j=1}^{n+1} (-1)^{n-j+1} \binom{n}{j-1} S_j^{[k]} = S_n^{[k-1]}. \qquad (2)$$

In fact, we may remember the recurrence relation [2]:

$$S_{n+1}^{[k]} = S_n^{[k-1]} + k\ S_n^{[k]}, \qquad (3)$$

then from (2) and (3):

$$-\sum_{j=k}^{n} (-1)^{n-j} \binom{n}{j-1} S_j^{[k]} + S_{n+1}^{[k]} = S_{n+1}^{[k]} - k\ S_n^{[k]},$$

which implies (1), q.e.d.

We can give an alternative proof for the identity (1) via the known Euler's expression [2]:

$$S_j^{[k]} = \frac{(-1)k}{k!} \sum_{r=0}^{k} (-1)^r \binom{k}{r} r^j, \qquad (4)$$

then $Q$ acquires the form:

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$$Q = \frac{(-1)^k}{k!}\sum_{r=0}^{k}(-1)^r\binom{k}{r}R \quad \text{such that} \quad R \equiv (-1)^n\sum_{j=1}^{n}(-1)^j\binom{n}{j-1}r^j, \tag{5}$$

but is immediate see that $R = r\,[r^n - (r-1)^n]$, hence:

$$Q = S_{n+1}^{[k]} - \frac{(-1)^{k-1}}{(k-1)!}\sum_{t=0}^{k-1}(-1)^t\binom{k-1}{t}t^n \overset{(4)}{=} S_{n+1}^{[k]} - S_n^{[k-1]} \overset{(3)}{=} k\ S_n^{[k]}, \quad \text{q.e.d.}$$

On the other hand, in [5] is the property:

$$\sum_{r=k}^{n} r\ S_n^{[r]}S_r^{[k]} = (-1)^{n+k+1}(n-k-1)! \geq \binom{n}{k-1}, \quad 1 \leq 1+k \leq n, \tag{6}$$

which was proposed by arithmetical experimentation, but we do not know the corresponding proof. This identity (6) is similar to the Akiyama-Tanigawa's relation [6, 7]:

$$\sum_{r=k}^{n} r\ S_n^{[r]}S_r^{(k)} = (-1)^{n+k}\binom{n}{k-1}, \quad 0 \leq k \leq n, \tag{7}$$

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