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Möbius Mapping via 3-Rotations and Lorentz Transformations

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Abstract: We exhibit that the Lorentz transformations and the 3-rotations generate Möbius mappings in the complex plane.

Key words: Lorentz transformations · Cayley-Klein parameters · Möbius mapping · Riemann sphere

INTRODUCTION

The stereographic projection establishes a correspondence between the points of the unit sphere and those of the Argand plane [1-3]:

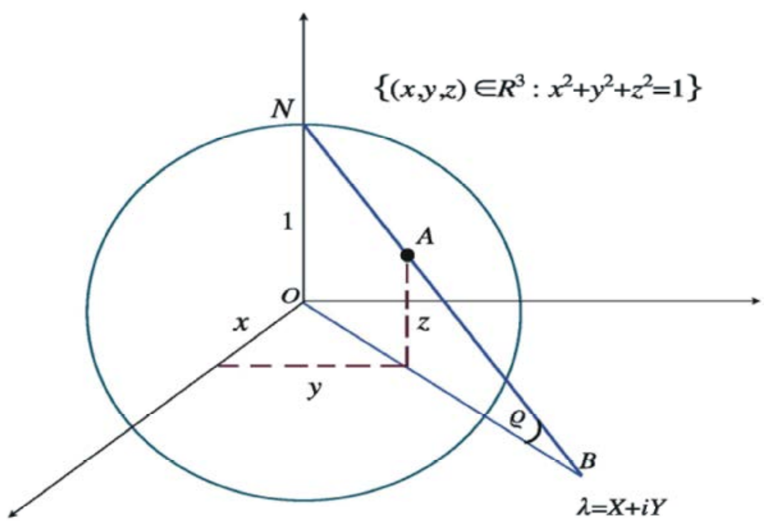

Fig. 1: The unit sphere is projected stereographically from its north pole to the complex plane through its equator.

thus it is easy to see that:

$$\lambda = X + iY = \frac{x+iy}{1-z}. \tag{1}$$

Under an arbitrary 3-rotation about the origin, each point of the sphere is mapped into another point on the sphere, hence via the stereographic projection a rotation determines a transformation of the complex onto itself. In fact, such three-dimensional rotation is given by [4, 5]:

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$$x' = \tfrac{1}{2}\left(\alpha^2 + \bar{\alpha}^2 - \beta^2 - \bar{\beta}^2\right)x - \tfrac{i}{2}\left(\alpha^2 - \bar{\alpha}^2 + \beta^2 - \bar{\beta}^2\right)y - \left(\alpha\beta + \bar{\alpha}\bar{\beta}\right)z,$$
$$y' = \tfrac{i}{2}\left(\alpha^2 - \bar{\alpha}^2 - \beta^2 + \bar{\beta}^2\right)x + \tfrac{1}{2}\left(\alpha^2 + \bar{\alpha}^2 + \beta^2 + \bar{\beta}^2\right)y - i\left(\alpha\beta - \bar{\alpha}\bar{\beta}\right)z,$$
$$z' = \left(\alpha\bar{\beta} + \bar{\alpha}\beta\right)x + i\left(\bar{\alpha}\beta - \alpha\bar{\beta}\right)y + \left(\alpha\bar{\alpha} - \beta\bar{\beta}\right)z, \qquad (2)$$

where the complex numbers α and α are the Cayley-Klein parameters [6] with the constraint $\alpha\overline{\alpha} + \beta\overline{\beta} = 1$ Therefore:

$$\lambda' = \frac{x' + iy'}{1 - z'} = \frac{\overline{\alpha}\lambda + \overline{\beta}}{-\beta\lambda + a}, \qquad (3)$$

which is a Möbius mapping [7] first obtained by Gauss [8, 9], that is, the most general rotation of the Riemann sphere can be expressed as a Möbius transformation of the form (3).

It is more convenient to employ $\zeta = \overline{\lambda}$. By contrast with λ, the variable ζ defines an orientation on the complex plane that coincides with that induced by the orientation of the sphere under the stereographic projection [2].

Now we consider a null cone with vertex in (0, 0, 0) for $t = 0$, then a slice of it at the time $t_0 > 0$ gives a sphere of radius $x^0 = ct_0$, which can be projected as in the Fig. 1, thus:

$$\zeta = \frac{x - iy}{x^0 - z}, \qquad (4)$$

and under an arbitrary Lorentz transformation [10, 11]:

$$\tilde{x} - i\tilde{y} = \bar{\delta}(\alpha\zeta + \beta)(x^0 - z) + \bar{\gamma}[\alpha(x^0 + z) + \beta(x + iy)],$$
$$\tilde{x}^0 - \tilde{z} = \bar{\delta}(\gamma\zeta + \delta)(x^0 - z) + \bar{\gamma}[\gamma(x^0 + z) + \delta(x + iy)], \qquad (5)$$

with the condition $\alpha\delta - \beta\gamma = 1$ we obtain again a Möbius mapping:

$$\tilde{\zeta} = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}, \qquad (6)$$

which has connection with special relativity [12]; Coxeter [13] comments that this connection was observed by Liebmann [14]. Thus, the complex mappings that correspond to the Lorentz rotations are the Möbius transformations with 6 degrees of freedom.

Remark 1: In Fig. 1 we have the relations:

$$NA = 2\sin\rho, \qquad NB = \frac{1}{\sin\rho} = \frac{2}{NA}. \qquad (7)$$

If we take the north pole of the Riemann sphere as centre of a sphere K with radius $\sqrt{2}$, then (7) means that the point B is the inversion in K of the point A. We may remember that the ‘inversion in a sphere’ also is called ‘transformation by reciprocal radii’ [7, 9, 15-17].

Remark 2: The null vector $(x^\mu) = (x^0, x, y, z)$ has associated the Cartan spinor [18]:

$$\left(X^{AB}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} x^0 + x^3 & x - iy \\ x + iy & x^0 - x^3 \end{pmatrix} = \frac{x^0 - x^3}{\sqrt{2}}\begin{pmatrix} \zeta\bar{\zeta} & \zeta \\ \bar{\zeta} & 1 \end{pmatrix}, \qquad (8)$$

where $x^0 > 0$ and $x^0 + z = (x + iy)\,\zeta$ and it is the product of two simple spinors:

$$\left(X^{AB}\right) = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}\begin{pmatrix} \xi^1 & \xi^2 \end{pmatrix} = \begin{pmatrix} \xi^1\xi^1 & \xi^1\xi^2 \\ \xi^2\xi^1 & \xi^2\xi^2 \end{pmatrix}, \qquad (9)$$

that is, $\xi^1\xi^1 = \frac{x^0 - x^3}{\sqrt{2}}\,\zeta\bar{\zeta}$ and $\xi^2\xi^1 = \frac{x^0 - x^3}{\sqrt{2}}\,\bar{\zeta}$, therefore:

$$\zeta = \frac{\xi^1}{\xi^2}, \qquad (10)$$

the stereographic projection is the quotient of the complex components of spinor ξ^A, in other words, ξ^1 and ξ^2 are homogeneous coordinates of ζ [9, 19]. Hence a pair of projective scalar ξ^1 and ξ^2 determine a unique point of the light-cone at each point of space-time; each value of the ratio ξ^1/ξ^2 determines a line of the cone. The homogeneous coordinates were invented by Möbius (1827) and Plücker (1830) [20].

Remark 3: The Riemann sphere permits a geometric interpretation of spin states for the electron and photon [21-23].

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