Computational and Applied Mathematical Sciences 31 (2): 21-22, 2018
ISSN 2222-1328

DOI: 10.5829/idosi.cams.2018.21.22

Some Properties of the Harmonic Numbers

[1]A. Zúñiga-Segundo, [2]J. López-Bonilla and [2]R. López-Vázquez

[1]Departamento de Física, ESFM, Instituto Politécnico Nacional, Edif. 9,
Col. Lindavista CP 07738, CDMX, México,
[2]ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso,
Col. Lindavista CP 07738, CDMX, México.

Abstract: We deduce identities involving the harmonic numbers H_n via their connection with the derivatives of binomial coefficients.

Key words: Binomial coefficients · Harmonic and Stirling numbers

INTRODUCTION

It is well known the property:

$$\frac{d}{dx}\binom{x+m}{n}=\binom{x+m}{n}\sum_{j=1}^{n}\frac{1}{j+x+m-n} \qquad (1)$$

in particular:

$$[\frac{d}{dx}\binom{x+m}{n}]_{x=n-m}=H_n, \quad [\frac{d}{dx}\binom{x}{n}]_{x=-1}=(-1)^{n+1}H_n, \qquad (2)$$

for the harmonic numbers [1]:

$$H_n=\sum_{r=1}^{n}\frac{1}{r}, \quad n\geq 1, \quad H_0=0. \qquad (3)$$

In Sec. 2 we employ (1) and (2) to deduce identities involving the quantities (3).

Harmonic Numbers:

We have the expression [2]:

$$x^n=\sum_{j=0}^{n} j!\binom{x}{j}S_n^{[j]}, \qquad (4)$$

where $S_n^{[j]}$ are Stirling numbers of the second kind [2- 4]; thus (2) and $[\frac{d}{dx}(4)]_{x=-1}$ imply:

$$\sum_{j=1}^{n}(-1)^j j!\, H_j S_n^{[j]}=n(-1)^n, \quad n\geq 1. \qquad (5)$$

We can verify (5), in fact [2, 5]:

$$H_j=\frac{(-1)^j}{j!}\sum_{q=1}^{j}(-1)^q q S_j^{(q)}, \qquad (6)$$

for the Stirling numbers of the first kind $S_n^{(m)}$, then:

$$\sum_{j=1}^{n}(-1)^j j!\, H_j S_n^{[j]}=\sum_{q=1}^{n}(-1)^q\, q\sum_{j=q}^{n}S_n^{[j]}S_j^{(q)}=(-1)^n n,$$

by the orthonormality of the Stirling numbers [2]; hence (5) and (6) are reciprocal relations.

Lanczos [6] used the binomial expansion of Gregory-Newton to obtain the identity:

$$\sum_{k=0}^{n}\binom{x}{k}\binom{n}{k}\frac{1}{(k+1)_m}=\frac{1}{(m+1)_m}\binom{x+m+n}{n}, \qquad (7)$$

where $(k+1)_m=\frac{(k+m)!}{k!}$; then (2) and $[\frac{d}{dx}(7)]_{x=-1}$ allow to deduce the formula:

$$\sum_{k=1}^{n}\frac{(-1)^{k+1}}{(k+1)_m}\binom{n}{k}H_k=\frac{1}{(m-1)!(m+n)}(H_{m+n-1}-H_{m-1}), \quad m\geq 1. \qquad (8)$$

We have the following expression of Graham-Knuth [7]:

Corresponding Author: Dr. J. López-Bonilla, ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México.

$$\sum_{k=0}^{n}\binom{x+k}{k}=\left(1+\frac{n}{x+1}\right)\binom{x+n}{n},\quad n\geq 0, \tag{9}$$

therefore (2) and $[\frac{d}{dx}(9)]_{x=0}$ imply the property [8]:

$$\sum_{k=0}^{n}H_k=(n+1)H_n-n,\quad n=0,1,2,..., \tag{10}$$

which is a particular case of the identity [5, 7-11]:

$$\sum_{k=m}^{n}\binom{k}{m}H_k=\binom{n+1}{m+1}\left(H_{n+1}-\frac{1}{m+1}\right), \tag{11}$$

for $m = 0$

In [2] we find the relation:

$$\sum_{k=1}^{n}(-1)^k\binom{x}{k}k=(-1)^n x\binom{x-2}{n-1},\quad n\geq 1, \tag{12}$$

thus (2) and $[\frac{d}{dx}(12)]_{x=-1}$ generate the result [12]:

$$\sum_{k=1}^{n}k\ H_k=\binom{n+1}{2}\left(H_{n+1}-\frac{1}{2}\right), \tag{13}$$

which is deductible from [2, 9]:

$$\sum_{k=1}^{n}k^m H_k=\sum_{j=1}^{m}\binom{n+1}{j+1}\left(H_{n+1}-\frac{1}{j+1}\right)j!S_m^{[j]}, \tag{14}$$

for $m = 1$

We know the expression:

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k+1}}{k\binom{x+k}{k}}=\sum_{k=1}^{n}\frac{1}{x+k}, \tag{15}$$

then $[\frac{d}{dx}(15)]_{x=0}$ and (2) allow to obtain the identity:

$$\sum_{k=1}^{n}\binom{n}{k}\frac{(-1)^{k+1}}{k}H_k=\sum_{k=1}^{n}\frac{1}{k^2}, \tag{16}$$

which can be verified directly via the relation:

$$H_k=\sum_{j=1}^{k}\binom{k}{j}\frac{(-1)^{j+1}}{j}, \tag{17}$$

consequence from (15) for $x = 0$.

REFERENCES

1. Carvajal-Gámez, B.E., J. López-Bonilla, R. López-Vázquez, 2017. On harmonic numbers, Prespacetime Journal, 8(4): 484-489.
2. Quaintance, J. and H.W. Gould, 2016. Combinatorial identities for Stirling numbers, World Scientific, Singapore.
3. López-Bonilla, J. and R. López-Vázquez, 2017. Harmonic numbers in terms of Stirling numbers of the second kind, Prespacetime Journal, 8(2): 233-234.
4. Barrera-Figueroa, V., J. López-Bonilla and R. López-Vázquez, 2017. On Stirling numbers of the second kind, Prespacetime Journal, 8(5): 572-574.
5. Benjamin, A.T., G.O. Preston and J.J. Quinn, 2002. A Stirling encounter with harmonic numbers, Maths. Mag., 75(2): 95-103.
6. Lanczos, C., 1996. Linear differential operators, SIAM, Philadelphia, USA (1996) Chap. 1
7. Graham, R.L., D.E. Knuth, O. Patashnik, 1994. Concrete Mathematics: a foundation for computer science, Addison-Wesley, Amsterdam.
8. Greene, D.H. and D.E. Knuth, 1990. Mathematics for the analysis of algorithms, Birkhäuser, Boston.
9. Spiess, J., 1990. Some identities involving harmonic numbers, Maths. of Comput., 55(192): 839-863.
10. Andrews, G.E. and K. Uchimura, 1995. Identities in combinatorics IV: differentiation and harmonic numbers, Utilitas Mathematica, 28: 265-269.
11. Choi, J. and H.M. Srivastava, 2011. Some summation formulas involving harmonic numbers and generalized harmonic numbers, Mathematical and Computer Modelling, 54: 2220-2234.
12. Stopple, J., 2003. A primer of analytic number theory, Cambridge University Press.