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On Two Theorems for the Weyl Tensor

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Abstract: We use the Newman-Penrose formalism to give elementary proofs of two theorems of Kozameh-Newman-Tod about contractions of the conformal tensor with skew-symmetric and symmetric real tensors of second order.

Key words: Weyl tensor · Newman-Penrose technique · Eigentensor · Null tetrad

INTRODUCTION

In Kozameh-Newman-Tod [1] we find the theorems [2]:

I: "Given a skew-symmetric real tensor $F^{\mu\nu}$, then the only solution of the equation:

$$C_{\mu\nu\alpha\beta} F^{\alpha\beta} = 0, \tag{1}$$

is $F_{\mu\nu} = 0$, provided:

$$J \equiv \frac{1}{96} S_{\mu\nu\alpha\beta} C^{\alpha\beta\gamma\tau} C_{\gamma\tau}{}^{\mu\nu} \neq 0, \tag{2}$$

where $C_{\mu\nu\alpha\beta}$ is the conformal tensor with [3]:

$$S_{\mu\nu\alpha\beta} = C_{\mu\nu\alpha\beta} + i * C_{\mu\nu\alpha\beta} \text{"}, \tag{3}$$

and:

II: "Given a symmetric and trace-free real tensor $E^{\mu\nu}$, then provided $J \neq 0$, the only solution of:

$$S_{\mu\nu\alpha\beta} E^{\mu\beta} = , \tag{4}$$

is $E_{\mu\nu} = 0''$,

Here we apply the Newman-Penrose (NP) technique [4-8] to give elementary proofs of both theorems.

Theorem I:
The condition (1) is equivalent to:

$$S_{\mu\nu\alpha\beta} \left(F^{\alpha\beta} + i * F^{\alpha\beta} \right) = 0, \tag{5}$$

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but we have the following NP expressions [4-7, 9, 10]:

$$S_{\mu\nu\alpha\beta} = 2\left[\psi_0 U_{\mu\nu} U_{\alpha\beta} + (U_{\mu\nu} M_{\alpha\beta} + M_{\mu\nu} U_{\alpha\beta}) + \psi_2 (M_{\mu\nu} M_{\alpha\beta} + V_{\mu\nu} U_{\alpha\beta} + U_{\mu\nu} V\alpha\beta)\right]$$
$$+\psi_3 (V_{\mu\nu} M_{\alpha\beta} + M_{\mu\nu} V_{\alpha\beta}) + \psi_4 V_{\mu\nu} V_{\alpha\beta}], \tag{6}$$

$$F^{\alpha\beta} + i * F^{\alpha\beta} = 2(\Phi_0 U^{\alpha\beta} + \Phi_1 M^{\alpha\beta} + \Phi_2 V^{\alpha\beta}), \tag{7}$$

$$J = \psi_2(\psi_1\psi_3 - \psi_2^2) + \psi_3(\psi_1\psi_2 - \psi_0\psi_3) + \psi_4(\psi_0\psi_2 - \psi_1^2), \tag{8}$$

thus (5) implies the homogeneous system:

$$\Phi_0\psi_2 - 2\Phi_1\psi_1 + \Phi_2\psi_0 = 0, \quad \Phi_0\psi_3 - 2\Phi_1\psi_2 + \Phi_2\psi_1 = 0, \quad \Phi_0\psi_4 - 2\Phi_1\psi_3 + \Phi_2\psi_2 = 0, \tag{9}$$

for the unknowns Φ_0, Φ_1 and Φ_2; it is immediate the determinant of (9):

$$\Delta = \begin{vmatrix} \psi_2 & -2\psi_1 & \psi_0 \\ \psi_3 & -2\psi_2 & \psi_1 \\ \psi_4 & -2\psi_3 & \psi_2 \end{vmatrix} = 2J \neq 0, \tag{10}$$

therefore the only solution of (9) is $\Phi_0 = \Phi_1 = \Phi_2 = 0$, hence $F_{\mu\nu}$, q.e.d.

Theorem II:

From [4, 5, 7] we have the NP expansion:

$$E_{\mu\nu} = 2\begin{bmatrix} \left(\overline{\Phi}_{12} m_\mu + \Phi_{12}\overline{m}_\mu\right) * l_\nu + \left(\overline{\Phi}_{01} m_\mu + \Phi_{01}\overline{m}_\mu\right) * n_\nu - \Phi_{11}\left(l_\mu * n_\nu + m_\mu * \overline{m}_\nu\right) - \\ \Phi_{22} l_\mu l_\nu - \Phi_{00} n_\mu n_\nu - \overline{\Phi}_{02} m_\mu m_\nu - \Phi_{02}\overline{m}_\mu \overline{m}_\nu \end{bmatrix}, \tag{11}$$

then (6) and (11) into (4) imply the following nine equations [three homogeneous systems]:

$$\psi_2\Phi_{00} - 2\psi_1\Phi_{10} + \psi_0\Phi_{20} = 0, \quad \psi_3\Phi_{00} - 2\psi_2\Phi_{10} + \psi_1\Phi_{20} = 0, \quad \psi_4\Phi_{00} - 2\psi_3\Phi_{10} + \psi_2\Phi_{20} = 0,$$

$$\psi_2\Phi_{02} - 2\psi_1\Phi_{12} + \psi_0\Phi_{22} = 0, \quad \psi_3\Phi_{02} - 2\psi_2\Phi_{12} + \psi_1\Phi_{22} = 0, \quad \psi_4\Phi_{02} - 2\psi_3\Phi_{12} + \psi_2\Phi_{22} = 0,$$

$$\psi_2\Phi_{01} - 2\psi_1\Phi_{11} + \psi_0\Phi_{21} = 0, \quad \psi_3\Phi_{01} - 2\psi_2\Phi_{11} + \psi_1\Phi_{21} = 0, \quad \psi_4\Phi_{01} - 2\psi_3\Phi_{11} + \psi_2\Phi_{21} = 0,$$

but the determinant of each system is given by (10), hence $J \neq 0$ indicates that the only solution is $\Phi_{ab} = 0$, $\forall\, a, b$ that is, $E_{\mu\nu} = 0$, q.e.d.

Thus our analysis shows that the Newman-Penrose formalism gives elementary proofs of the theorems of Kozameh-Newman-Tod [1, 2].

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