

An Application of the Tau Method

R. Cruz-Santiago, C. Hernández-Aguilar and J. López-Bonilla

ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 5,
 3er. Piso, Col. Lindavista CP 07738, CDMX, México

Abstract: We exhibit that the traditional formulation of the tau method is useful for the problems proposed by Issa-Adeniyi, hence it is not necessary to employ a complicate reformulation of this Lanczos process to construct polynomial solutions of ordinary differential equations.

Key words: Lanczos-Ortiz canonical polynomials • Tau technique

INTRODUCTION

Issa-Adeniyi [1] introduces a reformulation of the tau method [2-9] to obtain numerical solutions of certain class of problems in ordinary differential equations, for example, to solve:

$$y' - x^2 y = 0, \quad y(0) = 1, \quad (1)$$

and with their procedure they construct the following polynomial solution of 5th order:

$$y(x) = 1 + \frac{1}{26\,442\,910\,625} (42\,887\,122\,x - 1\,012\,427\,712\,x^2 + 13\,794\,322\,304\,x^3 - 9\,264\,926\,976\,x^4 + 6\,901\,011\,968\,x^5). \quad (2)$$

Here we exhibit that with the usual version of the tau method is possible to study (1) and to give an alternative polynomial solution of fifth order simpler than (2).

The Tau Method: The Lanczos method can be successfully applied to linear differential equations of arbitrary order, with the only condition that their coefficients have to be polynomials, with certain boundary conditions, in $[-1, 1]$ (strictly speaking this is not a restriction, since a change of scale can always be made) due to the fact that the Chebyshev polynomials $T_k(x)$ [10-13] vary uniformly in the whole interval; if we make the

analysis in $[0, 1]$ then the tau method remains unaltered, because it is only necessary to employ the modified Chebyshev polynomials $T_k^*(x)$ [14, 15] instead of the T_k . Let us consider the problem:

$$Dy(x) = 0, \quad (3)$$

such that D is a linear differential operator of order α , with the initial conditions:

$$y^{(k)}(0), \quad k = 0, 1, \dots, \alpha - 1 \quad (4)$$

In the next step the Lanczos-Ortiz canonical polynomials Q_m [2, 14, 16-19] are constructed, with the important clarification that m does not necessarily refer to the polynomial order:

$$DQ_m(x) = x^m + R_m(x), \quad m = 0, 1, 2, \dots \quad (5)$$

where the R_m are known as residual polynomials. It is important to note that there can exist certain values $m_1, m_2, \dots, m_s$, for which the prescription (5) does not work, namely, for which it is not possible to construct the pair Q_{m_j} and R_{m_j} verifying (5). In these cases, the Q_{m_j} are known as indefinite polynomials and it is convenient to introduce an ensemble S that contains such pathological values:

$$S = \{m_1, m_2, \dots, m_s\}, \quad (6)$$

on the other hand, all the residual polynomials are linear combinations of the different powers x^{m_j} , $j = 1, \dots, s$,

$$R_x(x) = C_k^{m_1} x^{m_1} + C_k^{m_2} x^{m_2} + \dots + C_k^{m_s} x^{m_s}, \quad k = 0, 1, \dots, \quad k \notin S, \quad (7)$$

note that in (7) we have $k \neq m_j$ because the R_{m_j} are indefinite.

Here it is accepted that the original problem (3) does not have exact polynomial solutions, which excludes the existence of multiple canonical polynomials. For instance, if two different polynomials Q_a and Q_b provide the same power of x , $Dq_a = Dq_b = x^p$, then $D(Q_a - Q_b) = 0$ and $Q_a - Q_b$ will be a polynomial solution of (3). It is possible to give the corresponding extension of the method in the case in which (3) allows exact polynomial solutions. Subsequently, Lanczos proposes to replace the zero in (3) for a small perturbation:

$$D\tilde{y}(x) = H_n(x), \quad (8)$$

where H_n is a n -degree polynomial and $\tilde{y}(x)$ is an exact polynomial solution subjected to the same boundary conditions (4):

$$D\tilde{y}(0) = y^{(k)}(0), \quad k = 0, \dots, a-1, \quad (9)$$

which in turn, is a good polynomial approximation for the problem (3), with an error uniformly distributed in $[-1, 1]$, we can achieve this last property if H_n is written in terms of Chebyshev's T_k :

$$H_n(x) = (\tau_0 + \tau_1 x + \dots + \tau_r x^r) T_{n-r}(x), \quad r = a + s - 1, \quad (10)$$

notice the presence of the $(r+1)$ parameters τ_j that the algorithm itself allows to determine and whose magnitudes are small because H_n should not deviate much from the zero of the right hand side of (3). Therefore, the quantity of parameters τ_j depends on the order α of the

differential operator D and also of the cardinality of S , i.e. of the number of the indefinite canonical polynomials. On the other hand, the Chebyshev polynomial that appears in (10) can be written in the following form:

$$T_{n-r}(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_{n-r} x^{n-r}, \quad (11)$$

its corresponding coefficients c_k are data that take part in different equations of the tau method.

Then, we can express the exact polynomial solution of (8) as:

$$\tilde{y}(x) = \sum_{m=0}^{n-r} c_m \sum_{\substack{i=0 \\ (m+i) \notin S}}^r \tau_i Q_{m+i}(x), \quad (12)$$

note that $(m+i)$ does not belong to (6) $\therefore (m+i) \neq m_j, j = 1, \dots, s$. If we impose in (12) the boundary conditions (9), we obtain the constraints:

$$\sum_{m=0}^{n-r} c_m \sum_{\substack{i=0 \\ (m+i) \neq m_j}}^r \tau_i Q_{m+i}^{(k)}(0) = y^{(k)}(0), \quad k = 0, 1, \dots, a-1, \quad (13)$$

and the substitution of (12) into (8) provides the relations:

$$\sum_{\substack{i=0 \\ (k+i) \in S}}^r \tau_i [\sum_{k=0}^{n-r} c_k x^{k+i} - \sum_{\substack{m=0 \\ (m+i) \notin S}}^{n-r} c_m R_{m+i}(x)] = 0. \quad (14)$$

In (13) there are α conditions, meanwhile (14) implies s constraints (because we should equal the coefficients of the different powers x^q to zero, whose q -values are all contained in S), making a total of $\alpha + s = r + 1$ algebraic equations to compute the $(r + 1)$ parameters τ_j , being therefore (12) completely determined. It should be emphasized that n is a datum, because it can be decided (depending on the problem under analysis) the order of the perturbation and therefore the tau process provides an exact solution for (8) and (9). Note that the order of $\tilde{y}(x)$ is not necessarily equal to n , in fact, this depends on the structure of the differential operator D .

If this process of Lanczos is applied to (1), that is, to (8) with $n = 7$, we obtain the quantities:

$\alpha = 1, \quad m_1 = 0, \quad m_2 = 1, \quad s = r = 2, \quad H_7 = (\tau_0 + \tau_1 x + \tau_2 x^2) T_5, \quad Q_0$ and Q_1 are indefinite,

$$T_5(x) = 5x - 20x^3 + 16x^5, \quad c_0 = c_2 = c_4 = 0, \quad c_1 = 5, \quad c_3 = -20, \quad c_5 = 16, \quad Q_2 = -1, \quad Q_3 = -x,$$

$$Q_4 = -x^2, \quad Q_5 = -x^3 - 3, \quad Q_6 = -x^4 - 4x, \quad Q_7 = -x^5 - 5x^2, \quad Q_8 = -x^6 - 6x^3 - 18, \quad (15)$$

$$Q_9 = -x^7 - 7x^4 - 28x, \quad R_2 = R_5 = R_8 = 0, \quad 28R_3 = 7R_6 = R_9 = -28, \quad 5R_4 = R_7 = -10x,$$

$$\tau_0 = -\frac{696}{32 \ 563}, \quad \tau_1 = -\frac{215}{32 \ 563}, \quad \tau_2 = -\frac{32}{32 \ 563},$$

with the exact solution of (8):

$$\tilde{y} = 1 + \frac{1}{32 \ 563} (-1 \ 740 x^2 + 10 \ 496 x^3 + 3 \ 440 x^4 + 512 x^5), \quad (16)$$

which is an approximate solution of (1) and simpler than (2).

A similar process can be applied to the several problems proposed by Issa-Adeniyi [1], without to modify the usual tau method.

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