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Some Identities for Stirling Numbers

[1]*J. Yaljá Montiel-Pérez,* [2]*J. López-Bonilla and* [2]*O. Salas-Torres*

[1]Centro De Investigación En Computación, Instituto Politécnico Nacional
[2]ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso,
Col. Lindavista CP 07738 CDMX, México

Abstract: We study the identities for Stirling numbers obtained by Wildon and Yuluklu *et al.*

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INTRODUCTION

Yuluklu-Simsek-Komatsu [1] deduced the identity:

$$A=\sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{j}2^{n-j}j!S_{n}^{(k)}S_{k}^{[j]}=(-1)^{n}n!, \tag{1}$$

where $S_n^{(k)}$ and $S_k^{[j]}$ are the Stirling numbers of the first and second kind, respectively [2]. In Sec. 2 we exhibit an elementary proof of (1) and we give an extension of it.

Wildon [3] used the technique of differentiation to obtain the following relations:

$$\sum_{k=0}^{n}\binom{n}{k}S_{k}^{[m]}=S_{n+1}^{[m+1]}, \tag{2}$$

$$\sum_{k=0}^{n}(-1)^{k}kS_{n}^{(k)}=-S_{n+1}^{(2)}, \tag{3}$$

$$\sum_{k=0}^{n}(-1)^{k}\binom{k}{m}S_{n}^{(k)}=(-1)^{m}S_{n+1}^{(m+1)}, \tag{4}$$

$$C=\sum_{k=0}^{n}\binom{n}{k}S_{k}^{[m]}B(n-k)=\sum_{r=0}^{n}\binom{r}{m}S_{n}^{[r]}, \tag{5}$$

with the participation of the Bell numbers [2, 4-6]:

$$B(q)\equiv\sum_{j=0}^{q}S_{q}^{[j]}. \tag{6}$$

In Sec. 3 we comment that the identities (2), (3) and (4) are known in the literature and we realize a simple demonstration of (5).

Yuluklu *et al.* Expression: We have the orthonormality of the Stirling numbers [2, 6]:

$$\sum_{k=j}^{n}S_{n}^{(k)}S_{k}^{[j]}=\delta_{jn}, \tag{7}$$

then:

$$A=\sum_{j=0}^{n}(-1)^{j}2^{n-j}j!\sum_{k=j}^{n}S_{n}^{(k)}S_{k}^{[j]}\overset{(7)}{=}(1)\ q.e.d.$$

Similarly:

$$D\equiv\sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{j-k}2^{n-j}j!S_{n}^{(k)}S_{k}^{[j]}=(-1)^{n}\sum_{j=0}^{n}(-1)^{j}2^{n-j}!L_{nj}, \tag{8}$$

with the presence of the Lah numbers [6-8]:

$$L_{n,j}\equiv\sum_{k=j}^{n}(-1)^{n-k}S_{n}^{(k)}S_{k}^{[j]}=\frac{n!}{j!}\binom{n-1}{j-1}, \tag{9}$$

thus from (8):

$$D=(-2)^{n-1}n!\sum_{q=0}^{n-1}\binom{n-1}{q}(-\frac{1}{2})^{q}=(-1)^{n+1}n!. \tag{10}$$

The identities (1) and (10) imply the result:

$$\sum_{k=0}^{n}\sum_{j=0}^{k}(-1)j-sk\ 2^{n-j}j!\ S_{n}^{(k)}\ S_{k}^{[j]}=\begin{cases}(-1)^{n}n!, & \varepsilon=0,\\ (-1)^{n+1}n!, & \varepsilon=1.\end{cases} \tag{11}$$

Correspnding Author: J. Yaljá Montiel-Pérez, Centro De Investigación En Computación, Instituto Politécnico Nacional.

Wildon's Relations: The property (2) is the equation (15.31) in [2], also see [9]. The relation (12.17) in [2] gives the following expression for the harmonic numbers:

$$H_n = \frac{(-1)^n}{n!} \sum_{k=0}^{n} (-1)^k k\, S_n^{(k)}, \tag{12}$$

besides, from [10] we have that:

$$H_n = \frac{(-1)^{n+1}}{n!} S_{n+1,}^{(2)} \tag{13}$$

hence (3) is consequence of (12) and (13). The identity (4) is deduced in [10].

From (9.25) in [2]:

$$D \equiv \sum_{k=0}^{n} \binom{n}{k} S_k^{[m]} S_{n-k}^{[j]} = \binom{m+j}{m} S_n^{[m+j]}, \tag{14}$$

which allows consider the left member of (5):

$$C \overset{(6)}{=} \sum_{j=0}^{n} D \overset{(14)}{=} \sum_{j=0}^{n} \binom{m+j}{m} S_n^{[m+j]} = \sum_{r=m}^{m+n} \binom{r}{m} S_n^{[r]},$$

equivalent to the right member of (5), *q.e.d.*

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