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On the Bernoulli's Differential Equation

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Abstract: It is well known that the Bernoulli's differential equation has the form $Y' + p(x)\ Y = q(x)Y^{\alpha}$ where α is a fixed real number. In this paper, under certain conditions, we give a generalization of this equation when we change α for an adequate function $r(x)$.

Key words: Bernoulli equation · Ordinary differential equation

INTRODUCTION

It is well known the Bernoulli's differential equation and how to solve it, which is given by [1-5]:

$$Y' + p(x)Y = q(x)Y^{a}, \tag{1}$$

where α is a fixed real number. In this work, we shall study the differential equation:

$$Y' + p(x)Y = q(x)Y^{r(x)}, \tag{2}$$

for certain functions $p(x)$, $q(x)$ and $r(x)$. For this, first we will see some types of differential equations with the structure:

$$Y' = \sum_{n=0}^{\infty} f_n(x)Y^n,$$

as we can see in Theorem 1. A solution $y = y(x)$ of the previous differential equation shall be represented by a series of powers, as in (4).

In reality, we will assume that all functions can be expanded in series of powers on an interval $I = (-t, t)$, with $t > 0$ and all the products between them will be convergent. Therefore, we will not bother to mention this.

Solution of a Differential Equation Expressed in Series of Powers: Let $\{f_n\}_{n\geq 0}$ and y functions. We write for each $n \geq 0$:

$$f_n(x) = \sum_{l=0}^{\infty} a_{n,l}x^l, \tag{3}$$

and

$$y = \sum_{m=0}^{\infty} b_m x^m, \tag{4}$$

Then, using the Cauchy product, we obtain:

$$y^n = \sum_{m=0}^{\infty} c_{n,m}x^m \tag{5}$$

where:

$$c_{n,m} = \sum_{s_1+\ldots+s_n=m} b_{s1}\ldots b_{s_n} \tag{6}$$

for all $n \geq 1$ and $m \geq 0$. We define:

$$c_{0,0} := 1 \text{ and } c_{0,m} := 0 \tag{7}$$

for all $m \geq 1$. Thus, we have:

Theorem 1: Under the above conditions and notations, we have that y is solution of the differential equation:

$$Y' = \sum_{n=0}^{\infty} f_n(x)Y^n \tag{8}$$

if and only if the coefficients of the representation of y as series of powers in (4) satisfy that $b_0 = y(0)$ and for each $m \geq 0$:

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$$b_{m+1} = \frac{1}{m+1}\sum_{k=0}^{m}\left(\sum_{n=0}^{\infty} a_{n,k}c_{n,m-k}\right). \quad (9)$$

Proof: We have that y is solution of (8) if and only if it holds:

$$\begin{aligned}\sum_{m=0}^{\infty}(m+1)b_{m+1}x^m = \quad y' = \sum_{n=0}^{\infty} f_n(x)y^n \\ =\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty} a_{n,l}x^l\right)\left(\sum_{m=0}^{\infty} c_{n,m}x^m\right) \\ =\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m} a_{n,k}c_{n,m-k}\right)x^m\right) \\ =\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} a_{n,k}c_{n,m-k}\right)\right)x^m \\ =\sum_{m=0}^{\infty}\left(\sum_{k=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{n,k}c_{n,m-k}\right)\right)x^m\end{aligned}$$

that is, if and only if we have:

$$\sum_{m=0}^{\infty}(m+1)b_{m+1}x^m = \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\left(\sum_{n=0}^{\infty} a_{n,k}c_{n,m-k}\right)\right)x^m.$$

Hence, y is solution of the differential equation (8) if and only if:

$$b_{m+1} = \frac{1}{m+1}\sum_{k=0}^{m}\left(\sum_{n=0}^{\infty} a_{n,k}c_{n,m-k}\right),$$

for all $m \geq 0$, where each $c_{n,m-k}$ is given as in (6). This completes the proof.

In particular, from (9) we have that $b_c = y(0)$,

$$b_1 = \sum_{n=0}^{\infty} a_{n,0}b_0^n, \quad (10)$$

and

$$b_2 = \frac{1}{2}\left(b_1\sum_{n=1}^{\infty} a_{n,0}b_0^{n-1} + \sum_{n=0}^{\infty} a_{n,1}b_0^n\right). \quad (11)$$

Corollary 2: Let f_0, f_1 and g functions and:

$$\sum_{n=2}^{\infty} a_n \quad (12)$$

a series absolutely convergent. We suppose that f_0 and f_1 are given as in (3) and we write:

$$g(x) = \sum_{l=0}^{\infty} u_l x^l. \quad (13)$$

Then, y is solution of the differential equation:

$$Y' = f_0(x) + f_1(x)Y + g(x)\sum_{n=2}^{\infty} a_n Y^n, \quad (14)$$

if and only if the coefficients of the representation of y as series of powers in (4) are given by $b_0 = y(0)$ and for each $m \geq 0$;

$$b_{m+1} = \frac{1}{m+1}\sum_{k=0}^{m}\left(a_{0,k}c_{0,m-k} + a_{1,k}c_{1,m-k} + \sum_{n=2}^{\infty} a_n u_k c_{n,m-k}\right). \quad (15)$$

Proof: We define for each $n \geq 2$, $f_n(x) := a_n g(x)$ for all $x \in (-$ it , $t)$, that is f_n is given as (3) such.
that:

$$a_{n,l} = a_n\, u_l, \quad (16)$$

for each $l \geq 0$ ($n \geq 2$). Then, applying the Theorem 1, we obtain (15) from (9).

Example 3: In the conditions and notations of Corollary 2, we have that y is solution of the differential equation:

$$Y' - f_0(x) + f_1(x)Y + g(x)\sum_{n=2}^{\infty}\frac{(-1)^n}{(n-1)^n}Y^n, \quad (17)$$

if and only if the coefficients of the representation of y in (4) are given by $b_0 = y(0)$ and for each $m \geq 0$:

$$b_{m+1} = \frac{1}{m+1}\sum_{k=0}^{m}\left(a_{0,k}c_{0,m-k} + a_{1,k}c_{1,m-k} + \sum_{n=2}^{\infty}\frac{(-1)^n}{(n-1)^n}u_k c_{n,m-k}\right). \quad (18)$$

A Particular Differential Equation: We conserve the conditions and notations of the Section 2. Also, we write:

$$h(x) = \sum_{l=0}^{\infty} v_l x^l, \quad (19)$$

thus, we have:

Theorem 4: We asume $|y(x) - 1| < 1$ for all $x \in (-t, t)$ Then y is solution of the differential equation:

$$Y' = f_0(x) + h(x)Y + g(x)Y\ln(Y), \tag{20}$$

if and only if the coefficients of y in its representation of series of powers in (4) are given by $b_0 = y(0)$ and, for each $m \geq 0$, b_{m+1} is given by the equation (15) of Corollary 2, where for each $n \geq 2$:

$$a_n = \sum_{m=1}^{\infty} a_{n-1,m} \tag{21}$$

with

$$a_{n,m} = \begin{cases} \dfrac{(-1)^{n+1}}{m}\dbinom{m}{n} & if\ 0 \leq n \leq m \\ 0 & if\ n > m \end{cases} \tag{22}$$

Proof: We have that:

$$\ln(y) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m}(y-1)^m, \tag{23}$$

because $|y(x) - 1| < 1$ for all $x \in (-t, t)$. Hence:

$$y\ln(y) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{n-1,m}\right)y^n, \tag{24}$$

such that:

$$a_{n,m} = \begin{cases} \dfrac{(-1)^{n+1}}{m}\dbinom{m}{n} & if\ 0 \leq n \leq m \\ 0 & if\ n > m \end{cases}$$

Then, the differential equation in (20) is equivalent to the following equation:

$$Y' f_0(x) + f_1(x)Y + g(x)\sum_{n=2}^{\infty}\left(\sum_{m=1}^{\infty} a_{n-1,m}\right)Y^n \tag{25}$$

where:

$$f_1(x) = h(x) + \left(\sum_{m=1}^{\infty} a_{n-1,m}\right)g(x), \tag{26}$$

therefore, the affirmation is followed from Corollary 2.

Bernoulli's Generalized Differential Equation: We consider Bernoulli's generalized differential equation (2), that is:

$$Y' + p(x)Y = q(x)Y^{r(x)},$$

where $r(x) \neq 1$ for all $x \in I$. We note that when $r(x)$ is constant on I, then the differential equation is the standard differential equation of Bernoulli.

Thus, under the substitution:

$$u = y^{1-r(x)} \tag{27}$$

we have the relationship:

$$\ln(u) = (1 - r(x)) \ln(y). \tag{28}$$

Using the equation (28) and Bernoulli's generalized differential equation (2), we obtain:

$$\frac{u'}{u} = \frac{d\ln(u)}{dx} = (1-r(x))q(x)\frac{1}{u} - p(x)(1-r(x)) - \frac{r'(x)}{1-r(x)}\ln(u), \tag{29}$$

or equivalently:

$$u' = (1-r(x))q(x) - p(x)(1-r(x))u - \frac{r'(x)}{1-r(x)}u\ln(u). \tag{30}$$

We note that the equation (30) establishes that u is solution of the differential equation given in Theorem 4, equation (20), where $f_0(x) = (1-r(x))\, q(x)$, $h(x) = -p(x)(1 - r(x))$ and $g(x) = -r'(x)/(1-r(x))$. Therefore, we know how is given the function u (Theorem 4). But, as $u = y^{1-r(x)}$ (equation (27)), then we have:

Theorem 5: We assume that $|r(x)| < 1$, for all $x \in (-r, r)$. The function y given in (4) is solution of the Bernoulli's generalized differential equation if and only if:

$$y = \sum_{n=0}^{\infty}\left(\frac{1}{n!}\left(\sum_{l=0}^{\infty} r(x)^l\right)^n\left(\sum_{k=0}^{\infty}\beta_k u^k\right)^n\right), \tag{31}$$

where for all $k \geq 0$:

$$\beta_k = \sum_{m=1}^{\infty} a_{k,m}, \tag{32}$$

and each $\alpha_{k,m}$ is given by the equation (22).

Proof: By the equation (27), we have:

$$y=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\ln(u)}{1-r(x)}\right)^{n}$$
$$=\sum_{n=0}^{\infty}\left(\frac{1}{n!(1-r(x))^{n}}\left(\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}(u-1)^{m}\right)^{n}\right)$$
$$=\sum_{n=0}^{\infty}\left(\frac{1}{n!(1-r(x))^{n}}\left(\sum_{m=1}^{\infty}\left(\sum_{k=0}^{m}\frac{(-1)^{k+1}}{m}\binom{m}{k}u^{k}\right)\right)^{n}\right)$$
$$=\sum_{n=0}^{\infty}\left(\frac{1}{n!(1-r(x))^{n}}\left(\sum_{m=1}^{\infty}\left(\sum_{k=0}^{\infty}a_{k,m}u^{k}\right)\right)^{n}\right)$$
$$=\sum_{n=0}^{\infty}\left(\frac{1}{n!(1-r(x))^{n}}\left(\sum_{k=0}^{\infty}\left(\sum_{m=1}^{\infty}a_{k,m}\right)u^{k}\right)^{n}\right)$$
$$=\sum_{n=0}^{\infty}\left(\frac{1}{n!(1-r(x))^{n}}\left(\sum_{k=0}^{\infty}\beta_{k}u^{k}\right)^{n}\right)$$
$$=\sum_{n=0}^{\infty}\left(\frac{\,}{n!}\left(\sum_{l=0}^{\infty}r(x)^{l}\right)^{n}\left(\sum_{k=0}^{\infty}\beta_{k}u^{k}\right)^{n}\right)$$

that is:

$$y=\sum_{n=0}^{\infty}\left(\frac{1}{n!}\left(\sum_{l=0}^{\infty}r(x)^{l}\right)^{n}\left(\sum_{k=0}^{\infty}\beta_{k}u^{k}\right)^{n}\right),$$

this completes the proof.

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