Computational and Applied Mathematical Sciences 31 (1): 01-03, 2018
ISSN 2222-1328

DOI: 10.5829/idosi.cams.2018.01.03

Local and Isometric Embedding of R_4 Into E_5

[1]J. Morales, [1]G. Ovando, [2]J. López-Bonilla and [2]S. Vidal-Beltrán

[1]CBI-Área de Física Atómica Molecular Aplicada, Univ., Autónoma Metropolitana-Azcapotzalco, Av. San Pablo 180, Col. Reynosa-Tamaulipas CP 02200, CDMX, México.
[2]ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México

Abstract: We establish conditions whose fulfillment implies that the spacetime admits embedding into E_5.

Key words: Local and isometric embedding ・ R_4 embedded into E_5 ・ Gauss-Codazzi equations ・ Thompson-Schrank theorem ・ Churchill-Plebañski classification

INTRODUCTION

R_4 accepts local and isometric embedding into E_5 if we can find the second fundamental form $b_{\mu\nu} = b_{\nu\mu}$ satisfying the equations [1-6]:

$$R_{\mu\nu\alpha\beta} = \varepsilon(b_{\mu\alpha}\, b_{\nu\beta} - b_{\mu\beta}\, b_{\nu\alpha}) \text{ Gauss} \quad (1.a)$$

$$b_{\mu\nu;\alpha} = b_{\mu\alpha;\nu} \text{ Codazzi} \quad (1.b)$$

where $\varepsilon = \pm 1$, $R_{\mu\nu\alpha\beta}$ is the curvature tensor and α indicates a covariant derivative.

In Sec. 2 we indicate intrinsic sufficient conditions which must be satisfied by the geometry of R_4 to guarantee the existence of $b_{\mu\nu}$. In Sec. 3 we use the Thompson-Schrank's theorem to show that $b_{\mu\nu}\, b^{\alpha\nu} \neq 0$ for the second fundamental form of R_4 embedded into E_s.

Spacetimes of Class One: We know that the Gauss equation implies the identity [7-9]:

$$p\, b_{\mu\nu} = \frac{K_2}{48}\, g_{\mu\nu} - \frac{1}{2}\, R_{\mu\alpha\beta\nu}\, G^{\alpha\beta}, \quad p = \frac{\varepsilon}{3}\, b_{\alpha\beta}\, G^{\alpha\beta}, \quad (2)$$

where $G_{\alpha\beta}$ is the Einstein tensor and K_2 is the Lanczos scalar [9-12]:

$$K_2 \equiv {}^*R^*_{\mu\nu\alpha\beta}\, R^{\mu\nu\alpha\beta} \quad (3)$$

such that [13]:

$$\det\left(b^{\mu}{}_{\nu}\right) = -\frac{1}{24} K_2 . \quad (4)$$

On the other hand, we have the Thomas theorem [14]:

"If det $(b^{\mu}{}_{\nu})$ then (1.a) implies (1.b)", (5)

which means that if R_4 has $K_2 \neq 0$ and we find $b_{\mu\nu}$ satisfying the Gauss equation, then the Codazzi condition is valid.

From (2) is immediate the relation:

$$p^2 = -\frac{\varepsilon}{6}\left(\frac{R}{24}\, K_2 + R_{\mu\alpha\beta\nu}\, G^{\mu\nu} G^{\alpha\beta}\right) \geq 0, \quad (6)$$

hence (2), (4), (5) and (6) allow to establish the following result:

"If the intrinsic geometry of the spacetime gives:

$$K_2 \neq 0, \qquad \left(\frac{R}{24}\, K_2 + R_{\mu\alpha\beta\nu}\, G^{\mu\nu}\, G^{\alpha\beta}\right) \neq 0, \quad (7)$$

then R_4 accepts local and isometric embedding into E_5".

In other words, (7) are intrinsic sufficient conditions for the corresponding embedding.

Let's remember that it is impossible the embedding of any empty R_4 into E_5 [13, 15-17], in fact, if $R^{\mu\nu} \equiv R^{\tau}{}_{\mu\nu}{}^{\tau}$ then from (1.a):

$$b_{\mu}{}^{\alpha}\, b^{\alpha\nu} = b\, b_{\mu\nu}, \qquad b^{\mu\alpha}\, b_{\mu\alpha} = b^2, \qquad b = b^{\nu}{}_{\nu}, \quad (8)$$

Corresponding Author: Dr. J. López-Bonilla, ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México.

therefore:

$$R^{\mu\nu\alpha\beta}\ R_{\mu\tau\alpha\gamma} = \varepsilon\ b^2\ R^{\nu}{}_{\gamma\tau}{}^{\beta}\ , \tag{9}$$

which allows to calculate the Synge's gravitational density [18, 19]:

$$F \equiv \frac{1}{2}(R_{\mu\nu}\ t^{\mu}\ t^{\nu})^2 + R^{\mu\nu\alpha\beta}\ R_{\mu\tau\alpha\gamma}\ t_{\nu}\ t_{\beta}\ t^{\tau}\ t^{\gamma} = 0, \tag{10}$$

where t^{μ} is an arbitrary timelike vector, hence R_4 is the Minkowski spacetime; thus any vacuum metric has an embedding class ≥ 2.

Thompson-schrank's Theorem: Thompson-Schrank [20] obtained the interesting result:

$$R_{\mu\nu}\ R^{\alpha\nu} = 0 \qquad \Longrightarrow \qquad R \equiv R^{\nu}{}_{\nu} = 0, \tag{11}$$

where $R_{\alpha\beta}$ and R are the Ricci tensor and the scalar curvature, respectively. In fact, (11) is equivalent to:

$$E_{\mu\nu}\ E^{\alpha\nu} + \frac{1}{2}R\ E_{\mu}{}^{\alpha} + \frac{1}{16}R^2\ \delta^{\alpha}_{\mu} = 0,$$
$$E_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{4}R\ g_{\alpha\beta}\ , \tag{12}$$

implies $R = 0$. If we compare (12) with each Churchill-Plebañski type [21-25], we find compatibility only with the types $[4T]_{[1]}$ and $[4N]_{[2]}$ for $R = 0$ thus (11) is immediate. Here we apply (11) to spacetimes of class one to deduce a constraint for the corresponding second fundamental form.

In fact, from (1.a):

$$R_{\mu}{}^{\alpha} = \varepsilon\left(b_{\mu\nu}\ b^{\alpha\nu} - b\ b_{\mu}{}^{\alpha}\right), \tag{13}$$

then we can see that the property:

$$b_{\mu\nu}\ b^{\alpha\nu} = 0\ , \tag{14}$$

is impossible for 4-spaces of class one because (14) and the Thompson-Schrank's theorem [20] imply $b = 0$, thus (13) gives $R_{\mu a} = 0$, but we know that no empty spacetime accepts embedding into E_5. Therefore:

$$b_{\alpha\beta}\ b^{\beta\gamma} \neq 0\ , \tag{15}$$

is a constraint for all spacetimes of class one.

REFERENCES

1. Lovelock, D. and H. Rund, 1989. Tensors, differential forms and variational principles, Dover, New York.
2. Caltenco, J.H., J. López-Bonilla and G. Ovando, 2001. Spacetime embedded into E_5, J. Bangladesh Acad. Sci. 25(1):: 95-97.
3. Stephani, H., D. Kramer, M. MacCallum, C. Hoenselaers and E. Herlt, 2003. Exact solutions of Einstein's field equations, Cambridge University Press (2003)
4. López-Bonilla, J., J. Morales, G. Ovando and E. Ramírez, 2006. Leverrier-Faddeev's algorithm applied to spacetimes of class one, Proc. Pakistan Acad. Sci. 43(1): 47-50.
5. Lam-Estrada, P., J. López-Bonilla and R. López-Vázquez, 2013. R_4 of class one, Transnational J. of Math. Analysis and Appl. 1(1): 5-8.
6. Hernández-Aguilar, C., A. Domínguez-Pacheco and J. López-Bonilla, 2017. Gauss-Codazzi equations in the Newman-Penrose formalism, Prespacetime Journal 8(5): 537-541.
7. López-Bonilla, J. and H.N. Núñez-Yépez, An identity for spacetimes embedded into E_5, Pramana J. Phys. , 46(3): 219-221.
8. López-Bonilla, J., J. Morales and G. Ovando, 2000. An identity for R_4 embedded into E_5, Indian J. Math. 42(3): 309-312.
9. Hernández-Aguilar, C., A. Domínguez-Pacheco and J. López-Bonilla, 2017. Embedding of R_n into E_{n+1}, Prespacetime Journal 8(5):: 533-536.
10. Lanczos, C., 1938. A remarkable property of the Riemann-Christoffel tensor in four dimensions, Ann. Math. 39 : 842-850.
11. López-Bonilla, J., J. Yaljá Montiel-Pérez and E. Ramírez, 2006. Lanczos invariant as an important element in Riemannian 4-spaces, Apeiron 13(2): 196-205.
12. López-Bonilla, J., H.N. Núñez-Yépez and A.L. Salas-Brito, 2016. On the Horndeski's formula for the Lanczos invariant, Prespacetime Journal 7(3): 509-511.
13. Fuentes, R., J. López-Bonilla and G. Ovando, 1989. Spacetimes of class one, Gen. Rel. Grav., 21(8):777-784.
14. Thomas, T.Y., 1936. Riemann spaces of class one and their characterizations, Acta Math., 67: 169-211.
15. Kasner, E., 1921. The impossibility of Einstein fields immersed in flat space of five dimensions, Am. J. Math. 43(2): 126-129.

16. Szekeres, P., 1996. Embedding properties of general relativistic manifolds, Nuovo Cimento A, 43(4): 1062-1076.
17. J. Caltenco, J. López-Bonilla, R. Peña-Rivero, The Synge scalar implies that empty non-flat spacetimes are not class one, Grav. & Cosm., 8(4): 318.
18. Synge, J.L., 1957. An invariant gravitational density, Proc. Roy. Irish Acad. A, 58: 29-39.
19. López-Bonilla, J., G. Ovando and J. Rivera, 2000. On some invariants of Synge, J. Bangladesh Acad. Sci., 24(2): 179-186.
20. Thompson, J. and G. Schrank, 1969. Algebraic classification of four-dimensional Riemann spaces, J. Math. Phys. 10(4): 766-770.
21. Churchill, R.V., 1932. Canonical forms for symmetric linear vector functions in pseudo-Euclidean space, Trans. Am. Math. Soc., 34: 784-794.
22. Plebañski, J., 1964. The algebraic structure of the tensor of matter, Acta Phys. Polon., 26: 963-1020.
24. Godina-Nava, J.J., J. López-Bonilla and A.L. Salas-Brito, 2016. Petrov classification of the Plebañski tensor, Open J. Appl. Theor. Maths. 2(1):: 21-26.
25. López-Bonilla, J., H.N. Núñez-Yépez, A.L. Salas-Brito and S. Vidal-Beltrán, 2017. Churchill-Plebañski and Petrov classifications for spacetimes of embedding class one, Prespacetime Journal 8(12): 1381-1386.