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An Expression for the Euler-Mascheroni's Constant

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Abstract: We employ the Lanczos approximation for the gamma function to deduce an expression for the Euler-Mascheroni's constant.

Key words: Euler-Mascheroni's constant · Gamma function

INTRODUCTION

Lanczos [1-3] obtained the following approximation for the gamma function [4-8]:

$$\Gamma(z+1)=2\lim_{k\to\infty}k^{z}\left[\frac{1}{2}-e^{-\frac{1}{k}}\frac{z}{z+1}+e^{-\frac{4}{k}}\frac{z(z-1)}{(z+1)(z+2)}-e^{-\frac{9}{k}}\frac{z(z-1)(z-2)}{(z+1)(z+2)(z+3)}+\ldots\right],\tag{1}$$

and $\mathrm{Re}\,z>-(k+\frac{1}{2})$ for a given value of k. The relation (1) allows to deduce an expression for the Euler-Mascheroni's constant [9-11] $\gamma =$ 0.5772 1566 4901 5328 6060 … because $\gamma = -\Gamma'(1)$, in fact:

$$\gamma=\lim_{k\to\infty}\left[2\sum_{r=1}^{\infty}\frac{1}{r}e^{-\frac{r^{2}}{k}}-Ln\,k\right].\tag{2}$$

For example, for $k >> 1$ we may use the approximation:

$$\gamma=2\sum_{r=1}^{k}\frac{1}{r}e-\frac{r^{2}}{k}-Ln\,k,\tag{3}$$

then from (3) for k = 2 000 and k = 1 000 000 we obtain the values γ = 0.5772 9900 0318... and γ = 0.5772 1583 1568..., respectively.

We note that the formula (1) can be written in the form:

$$\Gamma(z+1)=2\lim k^{z}[\frac{1}{2}+\sum_{r=1}^{\infty}(-1)^{r}e^{-\frac{r^{2}}{k}}\frac{\binom{z}{r}}{\binom{z+r}{r}}],\tag{4}$$

thus (2) is immediate because $\binom{0}{r}=0$ and $[\frac{d}{dz}\binom{z}{r}]_{z=0}=\frac{(-1)^{r-1}}{r}$ for $r \geq 1$.

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