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On Some Expressions for π

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Abstract: Some expressions of Chudnovsky and Gosper for π are written in terms of hypergeometric functions.

Key words: Hypergeometric functions · Mathematica

INTRODUCTION

In [1] we find the following expression of David and Gabriel Chudnovsky:

$$A_1 \equiv \sum_{m=0}^{\infty} \frac{2^{m+1}}{\binom{2m}{m}} = \pi + 4, \tag{1}$$

and in [2] are the Gosper's formulae:

$$A_2 \equiv \sum_{m=1}^{\infty} \frac{1}{m\binom{2m}{m}} = \frac{\pi\sqrt{3}}{9}, \qquad A_3 \equiv \sum_{m=1}^{\infty} \frac{3^m}{\binom{2m}{m}} = \frac{4\pi\sqrt{3}}{3} + 3,$$

$$A_4 \equiv \sum_{m=1}^{\infty} \frac{3^m}{m^2\binom{2m}{m}} = \frac{2\pi^2}{9}, \qquad A_4 \equiv \sum_{m=1}^{\infty} \frac{m}{\binom{2m}{m}} = \frac{2}{27}(\pi\sqrt{3} + 9). \tag{2}$$

Here we use the techniques of [3-5] to obtain the hypergeometric form of A_r, $r = 1, ..., 5$.

Deduction of (1) and (2) via Hypergeometric Functions and Mathematica: The relation (1) can be written as:

$$A_1 = 2\sum_{m=0}^{\infty} t_m, \qquad t_m = \frac{2^m}{\binom{2m}{m}}, \qquad t_0=1, \qquad \frac{t_{m+1}}{t_m} = \frac{(m+1)(m+1)}{\left(m+\frac{1}{2}\right)(m+1)}\ \frac{1}{2}, \tag{3}$$

then from [3-5] we deduce that:

$$A_1 = 2\,{}_2F_1\left(1,1;\frac{1}{2};\frac{1}{2}\right). \tag{4}$$

On the other hand, in [6] we find the following formula in terms of the gamma function:

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$$ {}_2F_1\left(a,b;\frac{a+b-1}{2};\frac{1}{2}\right)=\frac{2^{b-1}\Gamma\left(\frac{a+b-1}{2}\right)}{\Gamma(b)}\left[\frac{\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a-1}{2}\right)}+\frac{2\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{a}{2}\right)}+\frac{\Gamma\left(1+\frac{b}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)}\right], \tag{5} $$

which for $a = b = 1$ gives ${}_2F_1\left(1,1;\frac{1}{2};\frac{1}{2}\right)\frac{\pi}{2}+2,$, hence (4) implies the value (1) obtained by Gabriel and David Chudnovsky [1].

Similarly, from (2):

$$ A_2=\frac{1}{2}\sum_{m=0}^{\infty} t_m, \quad t_m=\frac{2(m+1)[m!]^2}{(2m+2)!}, \quad \frac{t_{m+1}}{t_m}=\frac{(m+1)(m+1)}{\left(m+\frac{3}{2}\right)(m+1)}\frac{1}{4} \quad \therefore \quad A_2=\frac{1}{2}\;{}_2F_1\left(1,1;\frac{3}{2};\frac{1}{4}\right), $$

$$ A_3=\frac{3}{2}\sum_{m=0}^{\infty} t_m, \quad t_m=\frac{2.3^m[(m+1)!]^2}{(2m+2)!}, \quad \frac{t_{m+1}}{t_m}=\frac{(m+1)(m+2)}{\left(m+\frac{3}{2}\right)(m+1)}\frac{3}{4} \quad \therefore \quad A_3=\frac{3}{2}\;{}_2F_1\left(1,2;\frac{3}{2};\frac{3}{4}\right), $$

$$ A_4=\frac{3}{2}\sum_{m=0}^{\infty} t_m, \quad t_m=\frac{2.3^m[m!]^2}{(2m+2)!}, \quad \frac{t_{m+1}}{t_m}=\frac{(m+1)(m+1)(m+1)}{(m+2)\left(m+\frac{3}{2}\right)(m+1)}\frac{3}{4} \quad \therefore \quad A_4=\frac{3}{2}\;{}_3F_2\left(1,1,1;2,\frac{3}{2};\frac{3}{4}\right), $$

$$ A_5=\frac{1}{2}\sum_{m=0}^{\infty} t_m, \quad t_m=\frac{2[(m+1)!]^2}{(2m+2)!}, \quad \frac{t_{m+1}}{t_m}=\frac{(m+2)(m+2)}{\left(m+\frac{3}{2}\right)(m+1)}\frac{1}{4} \quad \therefore \quad A_5=\frac{1}{2}\;{}_2F_1\left(2,2;\frac{3}{2};\frac{1}{4}\right), \tag{6} $$

and Mathematica gives the following values:

$$ {}_2F_1\left(1,1;\frac{3}{2};\frac{1}{4}\right)=\frac{2\pi\sqrt{3}}{9}, \qquad {}_2F_1\left(1,2;\frac{3}{2};\frac{3}{4}\right)=\frac{8\pi\sqrt{3}}{9}+2, \qquad {}_3F_2\left(1,1,1;2,\frac{3}{2};\frac{3}{4}\right)=\frac{4\pi^2}{27}, $$
$$ {}_2F_1\left(2,2;\frac{3}{2};\frac{1}{4}\right)=2.\ 1394\ 6638\ 4104\ 0966\ \ldots, \tag{7} $$

hence the expressions (6) imply the formulae (2) of Gosper [2].

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