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On Homogeneous Linear Differential Equations of 2th Order

[1]*J. Morales,* [1]*G. Ovando,* [2]*J.H. Caltenco and* [2]*J. López-Bonilla*

[1]CB1-Área de Física, AMA, Universidad Autónoma Metropolitana-Azcapotzalco,
Av. San Pablo 180, Col. Reynosa-Tamaulipas CP 02200, CDMX, México
[2]ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso,
Col. Lindavista CP 07738, CDMX, México

Abstract: The works [1-5] use an asymptotic iteration method to solve 2th order homogeneous linear differential equations, searching a special structure for one of the corresponding solutions. Here we exhibit that this type of solution must be polynomial and that its existence is controllated by a Riccati's equation.

Key words: Asymptotic iteration method · Riccati equation · Homogeneous linear differential equation

INTRODUCTION

The method indicated in [1-5] concerns to the homogeneous differential equation:

$$y'' - \lambda_0(x)\, y' - S_0(x)\, y = 0, \quad \lambda_0 \neq 0, \tag{1}$$

searching a solution with the structure:

$$y = \exp(-\int^x a(t)dt). \tag{2}$$

Then we must establish the conditions for the existence of $a(x)$ such that (2) verifies (1), which is realized in Sec. 2, thus a satisfies the Riccati equation [6-9] and the procedure from [1-5] only gives polynomial solutions for (1). In Sec. 3 we make applications to Hermite, Chebyshev and Laguerre equations.

Iterative Method: The proposal (2) implies:

$$y' = -ay, \tag{3.a}$$

$$y'' = (a^2 - a')y, \tag{3.b}$$

then (1) gives the following Riccati equation [6-9] as a compatibility condition:

$$a' - a^2 - \lambda_0\, a + S_0 = 0, \tag{4}$$

that is, $a(x)$ must satisfy (4) if (2) is solution of (1).

In (3.b) we employ (4) to obtain:

$$y'' = (S_0 - \lambda_0\, \alpha)y \tag{5}$$

thus in this equation is natural to ask the cancellation of its right member:

$$a = \frac{S_0}{\lambda_0}, \tag{6}$$

therefore:

$$y'' = 0 \tag{7.a}$$

$$y'' = 0 \tag{7.b}$$

then it is necessary apply $\frac{d}{dx}$ to (1) and use (5) to eliminate y'' in the final expression:

$$y''' - \lambda_1\, y' - S_1\, y = 0 \tag{8.a}$$

$$S_1 = S'_0 + S_0\, \lambda 0, \quad \lambda_1 = \lambda'_0 + \lambda^2{}_0 + S_0 \tag{8.b}$$

If (3.a) is applied in (8.a) we deduce the relation $y''' = (S_1 - \lambda_1\, a)y$ where we can employ (6) and (7.b) to obtain the constraint:

$$a = \frac{S_0}{\lambda_0} = \frac{S_1}{\lambda_1}, \tag{9}$$

Corresponding Author: J. López-Bonilla, ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México.

hence (8.b) and (9) imply the compatibility condition:

$$R_0 \equiv \lambda_0 S_0^{'} - S_0 \lambda_0^{'} - S_0^2 = 0. \tag{10}$$

The proposal (6) gives the solution (2) if $a = \frac{S_0}{\lambda 0}$ satisfies (4), in fact $\frac{d}{dx}\left(\frac{S_0}{\lambda_0}\right) - (\frac{S_0}{\lambda_0})^2$ in harmony with (10). In resumé, (2) is solution of (1) with α given by (6) if the functions S_0 and λ_0 verify (10) [which is equivalent to (9)].

From (4) and (6) we deduce that $a' - a^2 = 0$ whose solution is immediate:

$$a = \frac{S_0}{\lambda_0} = \frac{1}{q_1 - x}, \tag{11.a}$$

being q_1 an arbitrary constant, thus (2) implies:

$$y(x) = q_1 - x, \tag{11.b}$$

then $y' = -1$ & $y'' = 0$ in according with (3.a), (6), (7.a) and (11.a). It is important emphasize that in the case (9) the functions S_0 and λ_0 must satisfy (11.a), hence necessarily the solution of (1) shall have the form (11.b). The equation (1) is homogeneous, therefore any constant multiple of (11.b) also is a solution.
From (7.b), $y''' = 0$, thus:

$$y^{(4)} = 0, \tag{12}$$

which originates the application of $\frac{d}{dx}$ to (8.a), where we employ (1) to eliminate y'' in the final relation:

$$y^{(4)} - \lambda_2\, y' - S_2\, y = 0, \tag{13.a}$$

$$S_2 - S_1^{'} + S_0\, \lambda_1, \qquad \lambda_2 = \lambda_1^{'} + \lambda_0 \lambda_1 + S_1. \tag{13.b}$$

With (3.a) into (13.a) we obtain $y^{(4)} = (S_2 - \lambda_2\, a)y$ whose comparison with (12) implies:

$$a = \frac{S_2}{\lambda_0} = \frac{S_0}{\lambda_0}, \tag{14}$$

of easy verification because there we can substitute (13.b) to deduce that $\frac{d}{dx} R_0 + \lambda_0 R_0 = 0,$ which is correct by (10).

Successive derivatives allow construct similar equations to (8.a), (13.a), etc., therefore:

$$y^{(n+2)} - \lambda_n\, y' - S_n y = 0, \qquad y^{(n+2)} = (S_n - \lambda_n\, a) y \tag{15.a}$$

$$S_n = S_{n-1}^{'} + S_0 \lambda_{n-1}, \qquad \lambda_n = \lambda_{n-1}^{'} + \lambda_0 = \lambda_{n-1}^{'} + S_{n-1}, \tag{15.b}$$

and we notice that $a = \frac{S_0}{\lambda_0} = \frac{S_1}{\lambda_1}$ implies $a = \frac{S_2}{\lambda_2} = \frac{S_3}{\lambda_3} = \ldots,$ because $y^{(m)} = 0$, $m = 2,3, \ldots$ Instead (9), it may occur that:

$$a = \frac{S_1}{\lambda_1} = \frac{S_2}{\lambda_2}, \tag{16.a}$$

which it would imply:

$$a = \frac{S_3}{\lambda_3} = \frac{S_4}{\lambda_4} = \dots , \tag{16.b}$$

because $y^{(r)} = 0$, $r = 3,4, \dots$, with the fulfillment of the Riccati equation (4) and $y(x)$ an polynomial of order two:

$$y(x) = q_2 + q_1\, x - x^2, \tag{17.a}$$

where the constants q_1 and q_2 are obtained via:

$$a = -\frac{q_1 - 2x}{q_2 + q_1 x - x^2}, \tag{17.b}$$

and from (5) and (17.a) we deduce the relation:

$$q_2\, S_2 + q_1(\lambda_0 + S_0\, x) - (2\, \lambda_0 + S_0\, x)\, x = -2, \tag{18}$$

instead (11.a). We remember that S_0 and λ_0 are data; if we find constants q_1 and q_2 verifying (18) then (17.a) is solution of (1), besides also the expressions (16.a, b) and (17.b) are satisfied.
The previous results can be generalized, in fact, if we have n such that:

$$a = \frac{S_{n-1}}{\lambda_{n-1}} = \frac{S_n}{\lambda_n}, \tag{19}$$

then $y(x)$ given by (2) is a polynomial of degree n:

$$y(x) = q_n + q_{n-1}\, x + \dots + q_1 x^{n-1} - x^n, \tag{20.a}$$

and it is solution of (1), where the constants q_j are constructed with the expression:

$$a = -\frac{y'}{y} = -\frac{q_{n-1} + 2q_{n-2}x + \dots + (n-1)\, q_1 x^{n-2} - n\, x^{n-1}}{q_n + q_{n-1}x + \dots + q_1 x^{n-1} - x^n} \tag{20.b}$$

With (15.b) and (19) we can verify the fulfillment of (4), which permits guarantee that (2) is solution of (1). The condition (19) implies $a = \frac{S_{n+1}}{\lambda_{n+1}} = \frac{S_{n+2}}{\lambda_{n+2}} = \dots$, because $y^{(m)} = 0$, $m = n + 1, n + 2, \dots$ We emphasize that all solutions of (1), with the form (2) and the property (19), are polynomials, besides, the Riccati equation (4) is fundamental for the existence of a solution with this characteristics.

It is known [10] that if $y_1(x)$ is a solution of the homogeneous equation:

$$p(x)\, y'' + q(x) y' + r(x)\, y = 0, \tag{21.a}$$

then the other solution of (21.a) is given by:

$$y_2(x) = y_1(x) \int^x \frac{1}{[y_1(\eta)]^2} \exp\left(-\int^\eta \frac{q^{(t)}}{p^{(t)}} dt\right) d\eta. \tag{21.b}$$

The comparison of (1) and (21.a) gives $p = 1$, $q = -\lambda_0$, $r = -S_0$ & $y_1 = \exp(-\int^x a(t) dt)$, thus from (21.b) results:

$$y_2 = y_1 \int^x \exp[\int^\eta \left(\lambda_0(t) + 2a(t)\right) dt] d\eta, \tag{22}$$

which is the equation (2.12) in [1] (or the equation (8) in [2]). We indicate that $y_1(x)$ always is a polynomial due to (19), however, $y_2(x)$ is not necessarily a polynomial.

Hermite, Chebyshev and Laguerre Equations: Here we apply the iterative method exhibited in Sec. 2 to construct polynomial solutions:

a). Hermite's equation [11-14]:

$$y'' - 2x\, y' + 2k\, y = 0 \tag{23}$$

with $k = 1$ then $S_0 = -2$, $\lambda_0 = 2x$ and from (8.b) results that $S_1 = -4x$, $\lambda_1 = 4x^2$ besides (9) is verified because $a = \frac{S_0}{\lambda_0} = \frac{S_1}{\lambda_1} = -\frac{1}{x}$, whose comparison with (11.a) gives $q_1 = 0$ and (11.b) generates the corresponding solution of (1):

$$y(x) = x \;\propto\; H_1(x) = 2x. \tag{24.a}$$

If now we use $k = 2$, we have $S_0 = -4$, $\lambda_0 = 2x$, $S_1 = -8x$, $\lambda_1 = 4x^2 - 2$, $S_2 = -16x^2$, $\lambda_2 = 8x^3 - 4x$, where we employed (13.b), and (16.a) is satisfied because $a = \frac{S_1}{\lambda_1} = \frac{S_2}{\lambda_2} = -\frac{-2x}{\frac{1}{2} - x^2}$ with the structure (17.b), thus $q_1 = 0, q_2 = \frac{1}{2}$, and the solution (17.a) adopts the form:

$$y(x) = \frac{1}{2} - x^2 \qquad \propto \qquad H_2(x) = -2 + 4x^2, \;\text{ etc.} \tag{24.b}$$

In this manner, for each value of k we can construct the corresponding Hermite's polynomial.

b). Chebyshev's equation [12, 15, 16]:

$$y'' - \frac{x}{1-x^2} y' + \frac{k^2}{1-x^2} y = 0, \tag{25}$$

then for $S_0 = -\frac{9}{1-x^2}$, $\lambda_0 = \frac{x}{1-x^2}$, $S_1 = -\frac{27\,x}{(1-x^2)^2}$, $\lambda_1 = \frac{11\,x^2 - 8}{(1-x^2)^2}$, $S_2 = \frac{45(1-4x^2)}{(1-x^2)^3}$, $\lambda_2 = \frac{15\,(4x^3 - 3x)}{(1-x^2)^3}$, ...

such that $a = \frac{S_2}{\lambda_2} = \frac{S_3}{\lambda_3} = -\frac{\frac{s}{4} - 3x^2}{\frac{3}{4} - x^3}$, with $q_1 = 0,\ q_2 = \frac{3}{4},\ q_3 = 0,$, hence from (20.a):

$$y(x) = \frac{3}{4} x - x^3 \qquad \propto \qquad T_3(x) = -3x + 4x^3, \quad \text{etc.}$$

etc. (26)

Thus, for each value of k, this method gives a Chebyshev's polynomial.

Laguerre's equation [12, 13, 17, 18]:

$$y'' + \frac{1-x}{x} y' + \frac{k}{x} y = 0, \tag{27}$$

with $k = 3$, therefore:

$$S_0 = -\frac{3}{x}\ ,\ \lambda_0 = -\frac{1-x}{x}\ ,\ S_1 = \frac{6-3x}{x^2}\ ,\ \lambda_1 = \frac{x^2-5x+2}{x^2}\ ,\ S_2 = \frac{-18+18x-3x^2}{x^3}\ ,\ \lambda_2 = \frac{-6+18x-9x^2+x^3}{x^3}\ , \ldots$$

and (20.b) is satisfied because $a = \frac{S_2}{\lambda_2} = \frac{S_3}{\lambda_3} = -\frac{-18+18x-3x^2}{6-18x+9x^2-x^3}$, for $q_1 = 9$, $q_2 = -18$ and $q_3 = 6$:

$y(x) = 6 - 18x + 9x^2 - x^3 = 6\, L_3(x)$, etc. (28)

We conclude that this iterative method is useful to construct polynomial solutions of differential equations of second order, linear and homogeneous.

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