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Balanzario's Method to Obtain ζ(2n)

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Abstract: Balanzario shows a method to obtain $\zeta(2n)$, $n = 1, 2, \ldots$, without the explicit participation of Bernoulli polynomials $\phi_m(x)$ here we exhibit the relationship between his polynomials and the $\phi_m(x)$.

Key words: Bernoulli polynomials ・ Riemann zeta function

INTRODUCTION

Balanzario [1] obtains the Riemann zeta function [2, 3] at even integers via the result:

$$\zeta(2n) = -\frac{(-16)^n \pi^{2n}}{4^{n-2}} p_{2n+1}\left(\frac{1}{2}\right), \qquad n = 1, 2, \ldots \tag{1}$$

where:

$$p_2(x) = \frac{x^2}{2}, \quad p_3(x) = \frac{x^3}{6}, \quad p_4(x) = \frac{x^4}{4} - \frac{x^2}{48}, \quad p_5(x) = \frac{x^5}{120} - \frac{x^3}{144},$$
$$p_6(x) = \frac{x^6}{720} - \frac{x^4}{576} + \frac{x^2}{11520}, \quad p_7(x) = \frac{x^7}{5040} - \frac{x^5}{2880} + \frac{7x^3}{34560}, \ldots \tag{2}$$

which can be generated with $P_2(x)$ and the recurrence relations:

$$p_{2k+1}(x) = \int^x p_{2k} d\eta, \qquad p_{2k+2}(x) = \int^x p_{2k+1} d\eta - x^2 p_{2k+1}\left(\frac{1}{2}\right), \qquad k = 1, 2, \ldots \tag{3}$$

without constants of integration.
Thus, with (1) and (2) are immediate the values:

$$\zeta(2) = \frac{\pi^2}{6}, \qquad \zeta(4) = \frac{\pi^4}{90}, \qquad \zeta(6) = \frac{\pi^6}{945}, \ldots \tag{4}$$

in harmony with the relations reported in the literature [2, 3]; to find a closed expression for $\zeta(2n + 1)$, is an open problem. In the next Section we show how to write (1) and (2) in terms of the polynomials and numbers of Bernoulli.

Balanzario's Polynomials: We have the Bernoulli polynomials [5]:

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$$\phi_1(x) = x, \qquad \phi_2(x) = x^2 - x, \qquad \phi_3(x) = x^3 - \frac{3x^2}{2} + \frac{x}{2}, \qquad \phi_4(x) = x^4 - 2x^3 + x^2,$$
$$\phi_5(x) = x^5 - \frac{5x^4}{2} + \frac{5x^3}{3} - \frac{x}{6}, \qquad \phi_6(x) = x^6 - 3x^5 + \frac{5x^4}{2} - \frac{x^2}{2}, \ \ldots \tag{5}$$

with the properties:

$$\phi_1\left(\frac{1}{2}\right) = \frac{1}{2}, \qquad \phi_j(1) = 0, \qquad j = 2,3,\ \ldots; \qquad \phi_{2r+1}\left(\frac{1}{2}\right) = 0, \qquad r = 1,2,\ \ldots;\ \phi_k(0) = 0, \quad \forall k. \tag{6}$$

Then the polynomials (2) can be written in terms of (5), for example:

$$p_3(x) = \frac{1}{24}[\phi_3(2x) - 4\phi_3(x) + \phi_1(x)], \qquad p_5(x) = \frac{1}{5760}[3\phi_5(2x) - 48\phi_5(x) - 7\phi_1(x)], \ \text{etc.} \tag{7}$$

therefore:

$$p_3\left(\frac{1}{2}\right) = \frac{1}{48}, \qquad p_5\left(\frac{1}{2}\right) = -\frac{7}{11520}, \ \ldots \tag{8}$$

and thus (1) and (8) imply (4).
Besides, it is possible to prove that:

$$p_{2k+1}\left(\frac{1}{2}\right) = -\frac{1}{2}\tilde{B}_{2k}\left(\frac{1}{2}\right) = (-1)^k \frac{1-2^{2k-1}}{2^{2k}(2k)!}\tilde{B}_{2k}, \tag{9}$$

where $\tilde{B}_{2k}$ are the Faulhaber [6]-Bernoulli [7] numbers:

$$\tilde{B}_2 = \frac{1}{6}, \qquad \tilde{B}_4 = \tilde{B}_8 = \frac{1}{30}, \qquad \tilde{B}_6 = \frac{1}{42}, \ \ldots \tag{10}$$

with the Bernoulli-like polynomials [5]:

$$\tilde{B}_1(x) = x, \quad \tilde{B}_2(x) = \frac{1}{6}(3x^2 - 1), \quad \tilde{B}_3(x) = \frac{1}{6}(x^3 - x), \quad \tilde{B}_4(x) = \frac{1}{360}(15x^4 - 30x^2 + 7),$$
$$\tilde{B}_5(x) = \frac{1}{360}(3x^5 - 10x^3 + 7x), \qquad \tilde{B}_6(x) = \frac{1}{15120}(21x^6 - 105x^4 + 147x^2 - 31), \tag{11}$$

then $\tilde{B}_2\left(\frac{1}{2}\right) = -\frac{1}{24}$ and $\tilde{B}_4\left(\frac{1}{2}\right) = \frac{7}{5760}$ in accordance with (8) and (9). If into (1) we employ (9) appears the known relation [2, 3, 8]:

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}}{(2n)!}\tilde{B}_{2n}. \tag{12}$$

The expressions (7) and (9) show the relationship between the polynomials of Balanzario and Bernoulli.

REFERENCES

1. Balanzario, E.P., 2001. Elementary method to evaluate the Riemann zeta function at even integers, Miscelánea Matemática, 33: 31-41.
2. Titchmarsh, E.C., 1986. The theory of the Riemann zeta function, Clarendon Press, Oxford (1986).
3. Stopple, J., 2003. A primer of analytic number theory, Cambridge University Press (2003).
4. Stark, E.L., 1974. The series $\sum_{k=1}^{\infty}\frac{1}{k^m}, m = 2,3,4, \ldots,$ once more, Math. Mag., 47: 197-202.
5. Lanczos, C., 1966. Discourse on Fourier series, Oliver & Boyd, London (1966).
6. Faulhaber, J., 1931. Academiae algebrae, Ulm, Germany (1631).
7. Bernoulli, J., 1713. Ars Conjectandi, Thurneysen, Basel (1713).
8. Ahsan, Z., P. Lam-Estrada, J. López-Bonilla, R. López-Vázquez, 2013. On the Laforgia-Natalini's inequality for the Riemann zeta function, Transnational J. of Mathematical Analysis and Appls., 1(1): 1-4.