African Journal of Basic & Applied Sciences 3 (4): 111-115, 2011
ISSN 2079-2034

Quantum Thermodynamic Application in Fermionic-Bosonic and Supersymmetric Harmonic Oscillator

A.S. Arife

Department of Mathematical, Faculty of Science, Egypt

Abstract: In this paper, This article analysis the thermodynamic quantizes of some particle in simple harmonic oscillation like bosonic fermionic and supersymatric harmonic oscillator and treatment the fermions particle in box and derived the partition function, internal energy and entropy. These system both at low and high temperature. We are study the behavior of thermodynamic properties of a harmonically bound electron in magnetic field.

Key words: Variational iteration method · Laplace transform method · Eighth-order differential equation · Homotopy perturbation method · Numerical method

INTRODUCTION

Thermodynamic describes the behavior of system containing a large number N-particles. These system are characterized by their temperature, volume, number and type of particles the state of the system is then further described by it is the total energy and a variety of other parameters including the entropy, statistical thermodynamic is that branch of physics which studies macroscopic system from a microscopic or molecular point view, our goal is understanding and prediction of macroscopic properties from the properties of individual molecules making the system [1] plank's hypothesis; the quantization of the radiation oscillators in searching for a modification of the above treatment that would reduce the contribution of high frequencies to energy, plank was led to make assumption equivalent to the following: the energy of an oscillator of natural frequency v is restricted to integral multiples of basic unite $\hbar v$ [2].

We are analysis the quantum thermodynamic properties of some oscillator system like a Bosonic - Fermionic and Supersymatric harmonic oscillator. Since the harmonic oscillator is the model system. It is intuitive to study the basic quantum thermodynamic properties in classically speaking, the harmonic oscillator system a like particle moving in direct x connected by a spring to fixed point. The potential energy is:

$$V(x) = 1/2m\,\omega^2 x^2$$

In finally, particle in a box; assume v(x) in the time-independent Schrödinger equation to be zero inside one dimensional box of length a and infinite outside the box a free particle wave function is appropriate, the energy E_n implies to $E_n = n^2h^2 / 8ma^2$ and the wave function after normalization given by:

$$\Psi_n = \sqrt{2/a}Sin(n\pi/a)x$$

We try derived the two fundamental distribution law of statistical mechanics the Fermi-Dirac distribution and Bose- Einstein, applies to system whose N-body wave function is antisyamtic with respect to interchange of any two identical particle and the other distribution, applies to system whose N-body wave function is symmetric under such an interchange. We can use grand canonical ensemble formalism to obtain the thermodynamic and in section we analyses the problem of an electron the combined presence of magnetic field and a parabolic confinement potential.

Statistical Thermodynamic Quantities Thermodynamic Connection: The function Z; Ξ is the central statistical thermodynamic function of the canonical ensemble (N, V and T fixed) and is called the grand canonical ensemble function, canonical ensemble description is the most useful [4] one as in most particle case it is possible to control the temperature and not the energy of the system. Grand canonical ensemble partition function.

Corresponding Author: A.S. Arife, Department of Mathematical, Faculty of Science, Egypt.

$$\Xi = \sum_{N,j} e^{-E_{Nj}/KT} e^{\mu N/KT}$$

As the canonical partition function connection between thermodynamic and statistical thermodynamic for closed isothermal system if we can determine Ξ for a system we can calculate it thermodynamic properties.

$$\Xi = \sum_{N,j} \lambda^{N} e^{-\beta \varepsilon_n}$$

And the partition function Z the connection between thermodynamic and canonical ensemble, internal energy (average energy)

$$\bar{E} = KT^2 \left(\frac{\partial \ln Z}{\partial T} \right)_{N,V} \quad (4)$$

The entropy S

$$S = K \ln Z + KT \frac{\partial \ln Z}{\partial T}$$

And the Helmhotz free energy A in terms of Z by using [4-5] along with the fact that A=E-TS is given

$$A = -KT \ln Z$$

And the specific heat in constant volume heat capacity

$$C_v = \left(\partial \bar{E} / \partial T \right)_{N,V}$$

The Schrödinger equation in first and second quantization: We shall start or discuses by merely reformulating the Schrödinger equation in the Hamiltonian take the form [5].

$$H = \sum_{k} T(x_k) + \frac{1}{2} \sum_{k \neq i} V(x_k, x_i)$$

T is the Kinetic energy and V is the potential energy of interaction between particle and terms represent the interaction between every pair if particle, counted one which accounts for the factor of 1/2. The time-dependent Schrödinger equation by

$$i\hbar \frac{\partial}{\partial t} U(x_1, x_2, \dots\dots\dots, t_1) = HU(x_1, x_2, \dots\dots\dots, t_1)$$

We can now expand the many body wave functions as follows

$$U(x_1, x_2, \dots\dots\dots, t_1) = \sum_{E_1, E_2, \dots\dots} C(E_1, E_2, \dots\dots\dots) U(x_1) U(x_2) \dots\dots$$

The Schrödinger equation multiply by the expression

$$U_{E1}(x_1) \dots\dots\dots U_{en}(x_2)$$

Which is product of the adjoin wave function corresponding the fixed set of quantum number $E_1, \dots\dots\dots E_n$ the many particle wave function assumed to have the following properties $\psi(\dots\dots x_i, \dots\dots x_j, t) = \pm \psi(\dots\dots x_j, \dots\dots x_i, t)$

The interchange of the corresponding quantum numbers

$$C(\dots\dots E_i, \dots\dots\dots\dots E_j, t) = \pm C(\dots\dots E_j, \dots\dots\dots\dots E_i, t)$$

Particle that require the plus + sign are called bosons and temporarily concentrate on such system and the others sign are called fermions.

The many particle Hilbert space and creation and distribution operator, let to be complete and orthonormal which requires that theses states satisfy the condition

$$\langle n_1' n_2' \dots\dots\dots | n_1 n_2 \dots\dots \rangle = \delta_{n_1' n_1} \delta_{n_2' n_2} \dots\dots\dots\dots \; orthogonality$$

$$\sum_{n_1, n_2, \dots\dots} | n_1 n_2 \dots\dots\dots \rangle \langle n_1 n_2 \dots\dots\dots | = \hat{I} \dots\dots \; completness$$

Time-independent operator follows directly from the commutation rules for example.

$$[b_k, b_k'] = \delta_{kk'}$$

These are just the commutation rules for the creation and destruction operator of harmonic oscillator all properties of these operator

$$b_k^{\dagger} b_k | n_k \rangle = n_k | n_k \rangle n_k = 1, 2, 3, \dots\dots\dots\dots$$

$$b_k | n_k \rangle = \sqrt{n_k} | n_k - 1 \rangle \quad (16)$$

$$b_k^{\dagger} | n_k \rangle = \sqrt{n_k + 1} | n_k + 1 \rangle$$

In particular our desired occupation number basic states are simply the direct product of eigenstates of number operator for each mode

$$| n_1 n_2 \dots\dots\dots\dots n_\infty \rangle = | n_1 \rangle | n_2 \rangle \dots\dots\dots\dots | n_\infty \rangle$$

Occupation Number Representation of Fock Space, because the order does not matter, it is convenient only to count the particles in every state. To be able to reach all states of in occupation number representation we have to build linear combinations: An arbitrary vector $|\phi\rangle$ can be written as

$$|\phi\rangle = \sum_{n_1,.....n_k} \phi_{n_1,.........n_k} |n_1,.........n_k\rangle$$

Creation and Annihilation Operators Now it is easy to define operators creating and annihilating particles in a certain state

$$b_k |n_1,........n_k\rangle = \sqrt{n_k}\,|n_k,........,n_k - 1\rangle$$

$$b_k^{\dagger} |n_1,.........n_k\rangle = \sqrt{n_k + 1}\,|n_k,........,n_k + 1\rangle$$

$b_k^{\dagger}$ is called creation operator

Now let us consider a classical harmonic oscillator in one dimensional describe by a coordinate q and p. The Hamilton operator

$$H = \frac{1}{2M}(p^2 + M\,\omega^2 x^2)$$

by the operator $a^{\dagger}$ and a we can write the Hamilton

$$H = \hbar\omega\left(a_B^{\dagger} a_B + \frac{1}{2}\right)$$

The subscript B to identify operator and satisfy

$$\left[a_B, a_B^{\dagger} a_B\right] = a_B$$

$$\left[a_B^{\dagger}, a_B^{\dagger} a_B\right] = -a_B^{\dagger}$$

Also we can see $b_B^{\dagger}|n_B\rangle = \sqrt{n_B + 1}|n_B + 1\rangle$ and $b_B|n_B\rangle = \sqrt{n_B}|n_B - 1\rangle$ where the wave function

$$|n_B\rangle = \frac{(a_B^{\dagger})^n}{\sqrt{n_B!}}|0\rangle$$

Probability of a state with energy E_n is same as probability of oscillator having energy E_n with n the oscillator quantum Number the probability can define by

$$P_j = \frac{1}{Z} e^{-E_j / KT}$$

Where the partition function can be calculate

$$Z = Tr\left\{e^{-H_B / KT}\right\}$$

The partition function with density matrix $Z = Tr\left\{e^{-\beta H}\right\}$ and the quantum mechanics operator $\hat{M}$ corresponding to $\hat{M}$ and the operator $\hat{M}$ given by

$$\bar{M} = \frac{Tr\left(\hat{M} e^{-\beta H}\right)}{Tr\left(e^{-\beta H}\right)}$$

We can defined a new operator $\hat{\rho}$ by

$$\bar{M} = Tr\left(\hat{M}\hat{\rho}\right)$$

The Hamilton operator given by $\hat{H}$

$$\hat{H} = \hbar\omega\left(a_B^{\dagger} a_B + \frac{1}{2}\right)$$

Many problem and engineering science are modeled by ordinary differential equation; Nonlinear equation which one of the basic nonlinear eighth-order boundary value problems equation.

Bosonic Harmonic Oscillator: The observable remain the partition function.

$$Z = Tr\left(e^{-\beta H_B}\right) = \sum_n \langle n| e^{-\beta H_B} |n\rangle$$

$$Z = \left\{\sum_n \langle n| e^{-\beta a_B^{\dagger} a_B} |n\rangle\right\} e^{-\beta\hbar\omega/2}$$

And partition function given by

$$Z = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}} = \frac{1}{2\sinh(\hbar\omega/2KT)}$$

Where $\beta = 1/KT$ by hyperbolic function from the partition function we can derive all thermodynamic quantities and are given by

Free energy

$$A = -Ktln2 + KT\ln(\sinh(\hbar\omega/2KT))$$

Internal energy

$$E = \frac{\hbar\omega}{2}\coth(\hbar\omega/2KT)$$

Entropy

$$\frac{S}{K}=\frac{\hbar\omega}{2KT}\coth(\hbar\omega/2KT)-\ln\left(2\sinh\left(\hbar\omega/2KT\right)\right)$$

Specific heat

$$\frac{C_v}{K}=\left(\frac{\hbar\omega}{2KT}\right)^2\cos ech^2\left(\hbar\omega/2KT\right)$$

All thermodynamic quantities of a simple bosonic harmonic oscillator.

Fermionic Harmonic Oscillator: In order introduce the fermionic harmonic oscillator; let us start by briefly the main formula for the bosonic case, in the operator description the basic commutation relation the Hamilton, the energy eigenstate and the completeness relation can be expressed as

$$[\hat{a},\hat{a}]=0 \quad \left[\hat{a}^{\dagger},\hat{a}^{\dagger}\right]=0 \quad \left[\hat{a},\hat{a}^{\dagger}\right]=1$$

The completeness

$$\sum_n |n\rangle\langle n|=\int dx|x\rangle\langle x|=\frac{1}{2\pi\hbar}\int dp\,|p\rangle\langle p|$$

In the fermionic case we replace the algebra of equation Hamilton

$$\{\hat{a},\hat{a}\}=0 \quad \left\{\hat{a},\hat{a}^{\dagger}\right\}=1$$

Consider next the Hilbert space the analogue $\hat{H}$ we defined the vacuum state $|0\rangle$

$$\hat{a}|0\rangle=0$$

Then we defined a one -particle state of $|1\rangle$

$$|1\rangle = a^{\dagger}|0\rangle$$

By the last equation and multiply the operator

$$\hat{a}|1\rangle=\hat{a}\hat{a}^{\dagger}|0\rangle=\left(1-\hat{a}^{\dagger}\hat{a}\right)|0\rangle=|0\rangle$$

The Hamilton the analogue of $\hat{H}$ is an operator acting in this space and can be defined as

$$\hat{H}=\hbar\omega\left(a_B^{\dagger}a_B-\frac{1}{2}\right)$$

The observable remain the partition function

$$Z=Tr\left(e^{-\beta H_F}\right)=\langle 0|e^{-\beta H_B}|0\rangle+\langle 1|e^{-\beta H_B}|1\rangle$$

$$Z=\left\{\langle 0||0\rangle+\langle 1|e^{-\beta\hbar\omega}|1\rangle\right\}e^{-\beta\hbar\omega/2}$$

$$Z=\left(1+e^{-\beta\hbar\omega}\right)e^{-\beta\hbar\omega/2}=2\cosh\left(\hbar\omega/2KT\right)$$

We can derived all thermodynamic quantities

Free energy

$$F=-KT\ln\left(2\cosh(\hbar\omega/2KT)\right)$$

Internal energy (average energy)

$$\bar{E}=-\frac{\hbar\omega}{2}\tanh\left(\hbar\omega/2KT\right)$$

Entropy

$$\frac{S}{K}=-\frac{\hbar\omega}{2KT}\tanh\ (\hbar\omega/2KT)+\ln\left(2\cosh\left(\hbar\omega/2KT\right)\right)$$

The specific heat

$$\frac{C_v}{K}=\left(\frac{\hbar\omega}{2KT}\right)^2\sec h^2\left(\hbar\omega/2KT\right)$$

Supersymmetric Harmonic Oscillator: Supersymmetric oscillator is a simple toy mode in quantum field theory and it is a combination of bosonic and femionic oscillator in the same ω product state are denoted by $|n_1 n_2 \ldots\ldots\ldots\, n_\infty\rangle = |n_1\rangle|n_2\rangle\ldots\ldots\ldots|\,n_\infty\rangle$

In femionic and bosonic H_S is given by

$$H_S=H_B+H_F$$
$$N_B|n_B\rangle=n_B|n_B\rangle$$
$$N_F|n_F\rangle=n_F|n_F\rangle$$
$$H_S|n_F,n_B\rangle=\hbar\omega(n_F+n_B)|n_F\rangle|n_B\rangle$$

For the composite system of the bosonic and fermionic $\omega = \omega_B = \omega_F$ we get the supersymmetric oscillator is

$$E_n = \hbar\omega(n_F + n_B) \qquad (52)$$

The partition function

$$Z = Tr\left(e^{-\beta H_B}\right)$$

$$Z = Tr\left(e^{-\beta H_B}\right) = \left\{ \sum_{n_F = n_B = 0} \langle n_F, n_B | \left(e^{-\beta\hbar\omega}\right) | n_B, n_F \rangle \right\}$$

$$Z = \langle 0 \| 0 \rangle + \sum_{n_B} \langle n_B | \left(e^{-n_B \beta\hbar\omega}\right) | n_B \rangle + \sum_{n_F} \langle n_F | \left(e^{-n_B \beta\hbar\omega}\right) | n_F \rangle$$

$$Z = \frac{e^{\hbar\omega/KT} + 1}{e^{\hbar\omega/KT} - 1} = \coth(\hbar\omega/KT)$$

We can derive all thermodynamic quantities
Free energy

$$A = -KT \ln \coth(\hbar\omega / KT)$$

Internal energy

$$\bar{E} = \frac{\hbar\omega}{\sinh(\hbar\omega/KT)}$$

CONCLUSION

In this paper, we analyze the thermodynamic quantities of some simple oscillator systems like bosonic harmonic oscilla-tor, fermionic harmonic oscillator and a super symmetric har-monic oscillator (which is a combination of bosonic and ferm-ionic oscillators) and study the behavior of the thermodynamic properties.

REFERENCES

1. Das, A., 1993. Field theory: A Path Integral Approach, (WorldScientist, Publishing, Singapore).
2. Pauli, W.Z., 1927. Phys., 41: 81.
3. Nernst, W., 1911.Nachrr. Ges. Wiss. Goettingen, Math. Phys, K1. 6, 1 (1906); also in Sitzungsber. K. Preuss. Akad. Wiss., 13: 311.
4. Mayer, J.E. and M.G. Mayer, 1940. Statistical Mechanics (John Wiley, New York).
5. Gibbs, J.W., 1961. The Scientist papers of J. Williard Gibbs Vol 1Thermodynamics, Dover.
6. Boltzmann, L., 1964. Lectures on Gas Theory (University of California Press, Los Angeles) pp: 3.
7. Schrödinger, E ., 1948. Statistical Thermodynamics. A course of seminar lectures, del. in Jan. March 1944 (Cambridge University Press, Cambridge) pp: 4.
8. Nielsen, M. and I. Chuang, 2000. Quantum Computation and Quantum Information (Cambridge University Press, Cambridge) pp: 5.
9. Landau, L. and E. Lifshitz, 1980. Statistical Physics, Part 1, Course of Theoretical Physics, Vol. 5, 3rd edn. (Pergamon Press, Oxford) pp: 4.
10. Slusher, R.E., A.F.J. Levi, U. Mohideen, S.L. McCall, S.J. Pearton and R.A. Logan, 1993. Threshold characteristics of semiconductor microdisk lasers, Appl. Phys. Lett., 63: 1310.