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Differential Pencils with a Turning Point

A. Neamaty and Y. Khalili

Department of Mathematics, University of Mazandaran, Babolsar, Iran

Abstract: In this paper we investigate boundary value problems for second-order differential pencil on the half-line having turning point. Using of the asymptotic forms of solutions of the differential equation on the half-line, we get the asymptotic distribution of the eigenvalues.

Key words: Asymptotic form · Sturm-Liouville · Turning point · Eigenvalues.
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INTRODUCTION

Differential equations with nonlinear dependence on the spectral parameter and with turning points arise in various problems of mathematics as well as in applications [1]. Some aspects of the inverse problem for differential pencils without turning points were studied in [2]. Indefinite differential equations with turning points produce significant qualitative modification in the investigation of the inverse problem. For classical Sturm-Liouville operators with turning points in the finite interval have been studied fairly completely in [3]. Some aspects of the inverse problem for differential pencils without turning points were studied in [4]. Here we investigate boundary value problem for differential pencils with turning points.
We consider the differential equation

$$y''(x)+(\rho^2R(x)+i\rho q_1(x)+q_0(x))y(x)=0\ ,\quad x\geq 0, \qquad (1)$$

on the half-line with nonlinear dependence on the spectral parameter ρ. Let $\omega>0$ and let

$$R(x)=\begin{cases}x-1, & x\geq a\geq 1,\\ -\omega^2, & 0\leq x<a,\end{cases} \qquad (2)$$

i.e. the sign of the weight-function changes in interior point, which is called the turning point. The function $q_j(x)$, J = 1,2 are complex-valued, $q_j(x)$ is absolutely continuous and $(1+x)q_j^{(l)}\in L(0,\infty)$ for $0\leq l\leq j\leq 1$.

In this paper, we will study the solutions and eigenvalues of the boundary value problem with spectral boundary condition. In section 2, we determine the asymptotic form of the solutions of (1) and using these asymptotic estimates, derive characteristic function and eigenvalues.

Properties of the Spectral Characteristics: We consider the boundary value problem L for Eq.(1) on the half-line with the boundary condition

$$U(y):=y'(0)+(\beta_1\rho+\beta_0)y(0)=0, \qquad (3)$$

where the coefficients β_1 and β_0 are complex numbers and $\beta_1 \neq \pm \omega$. Denote $\Pi\pm:=\{\rho:\pm Im\rho>0\}$, $\Pi_0:=\{\rho: Im\rho=0\}$. By well-known method [5, 6], we have the following theorem:

Theorem 1: Equation (1) has an unique solution $y = e(x,\rho)$, $\rho\in\Pi\pm$, $x\geq a$ with the following properties :

- For each fixed $x\geq a$, the functions $e^{(v)}(x,\rho)$, $v = 0,1$ are holomorphic for $\rho\in\Pi_+$ and $\rho\in\Pi_-$ (i.e. they are piecewise holomorphic).
- The functions $e^{(v)}(x,\rho)$, $v = 0,1$ are continuous for $x\geq a$, $\rho\in\Pi_+$ and $\rho\in\Pi_-$ (we differ the sides of the cut Π_0). In the other words, for real ρ, there exist the finite limits

$$e_\pm^{(v)}(x,\rho)=\lim_{z\to\rho,\, z\in\Pi_\pm} e^{(v)}(x,z).$$

Moreover, the functions $e^{(v)}(x,\rho)$, $v = 0,1$ are continuously differentiable with respect to $\rho\in\overline{\Pi}_+\setminus\{0\}$ and $\rho\in\overline{\Pi}_-\setminus\{0\}$.

Corresponding Author: A. Neamaty, Department of Mathematics, University of Mazandaran, Babolsar, Iran.
E-mailnamaty@umz.ac.ir.

For $x \to \infty, \rho \in \overline{\Pi}_{\pm} \setminus \{0\}, \nu = 0,1,$

$$e^{(\nu)}(x,\rho) = (\pm i\rho)^{\nu} R(x)^{\nu - \frac{1}{2}} exp(\pm(i\rho x - Q(x)))(1 + o(1)), \quad (4)$$

where

$$Q(x) = \frac{1}{2}\int_0^x q_1(t)dt. \quad (5)$$

For $|\rho| \to \infty, \rho \in \overline{\Pi}_{\pm}, \nu = 0,1$ uniformly in $x \geq a$

$$e^{(\nu)}(x,\rho) = (\pm i\rho)^{\nu} R(x)^{\nu - \frac{1}{2}} exp(\pm(i\rho x - Q(x)))[1], \quad (6)$$

where $[1] := 1 + O(\rho^{-1})$.

The function $e(x,\rho)$ is called the Jost-type solution for Eq.(1).

We extend $e(x,\rho)$ to the segment $[0, a]$ as a solution of E.q(1) which is smooth for $x \geq 0$, i.e.

$$e^{(\nu)}(a-0,\rho) = e^{(\nu)}(a+0,\rho), \nu = 0,1. \quad (7)$$

Then the properties 1-2 remain true for $x \geq 0$.

Let $\varphi(x,\rho)$ and $S(x,\rho)$ be solutions of Eq.(1) under the conditions $\varphi(x,\rho) = 1$, $U(\varphi) = 0$, $S(x,\rho) = 0$, $S'(x,\rho) = 1$ For each fixed $x \geq 0$, the functions $\varphi^{(\nu)}(x,\rho)$ and $S^{(\nu)}(x,\rho)$, $\nu = 0,1$ are entire in ρ.

Lemma 2: The following asymptotic formulae are valid

1. For $|\rho| \to \infty$, $m = 0,1$ uniformly in $x \in [0, a]$

$$\varphi^{(m)}(x,\rho) = \frac{(\omega\rho)^m}{2\omega}((-1)^m(\omega + \beta_1)exp(-\omega\rho x + i\frac{Q(x)}{\omega})[1] + (\omega - \beta_1)exp(\omega\rho x - i\frac{Q(x)}{\omega})[1]),$$

2. For $|\rho| \to \infty$, $m = 0,1$ uniformly in $x \in [0, \infty]$

$$\begin{aligned}
\varphi^{(m)}(x,\rho) = {} & (\frac{1+\beta_1}{-4}(1+i(a-1)^{\frac{1}{2}})exp(-\rho a(1+i(a-1)^{\frac{1}{2}}) + iQ(a)(1-i(a-1)^{\frac{-1}{2}}))[1] \\
& + \frac{1-\beta_1}{4}(1-i(a-1)^{\frac{1}{2}})exp(\rho a(1-i(a-1)^{\frac{1}{2}}) - iQ(a)(1+i(a-1)^{\frac{-1}{2}}))[1]) \\
& \times (i\rho(x-1)^{\frac{1}{2}})^m exp(i\rho(x-1)^{\frac{1}{2}}x - (x-1)^{\frac{-1}{2}}Q(x)) + (\frac{1+\beta_1}{-4}(1+i(a-1)^{\frac{1}{2}}) \\
& \times exp(-\rho a(1-i(a-1)^{\frac{1}{2}}) + iQ(a)(1+i(a-1)^{\frac{-1}{2}}))[1] + \frac{1+\beta_1}{4}(1-i(a-1)^{\frac{1}{2}}) \\
& \times exp(\rho a(1+i(a-1)^{\frac{1}{2}}) - iQ(a)(1-i(a-1)^{\frac{-1}{2}}))[1])(-i\rho(x-1)^{\frac{1}{2}})^m \\
& \times exp(-i\rho(x-1)^{\frac{1}{2}}x + (x-1)^{\frac{-1}{2}}Q(x)).
\end{aligned}$$

Proof: Denote $\Pi^1_{\pm} := \{\rho : \pm Re\rho > 0\}$. It is known [5, 6] that for $x \geq a, m = 0,1$ $\rho \in \overline{\Pi}_{\pm}, |\rho| \to \infty,$ there exists a fundamental systems of solutions $\{Y_k(x\ \rho)\}_{k=1,2}$ of Eq.(1) of the form

$$Y_k^{(m)}(x,\rho) = ((-1)^{k-1}i\rho(x-1)^{\frac{1}{2}})^m exp((-1)^{k-1}(i\rho x(x-1)^{\frac{1}{2}} - (x-1)^{\frac{-1}{2}}Q(x)))[1]. \quad (8)$$

Similarly for $x \in [0,a], \rho \in \overline{\Pi}^1_{\pm}, |\rho| \to \infty,$ there exists a fundamental systems of solutions $\{Y_k(x\ \rho)\}_{k=1,2}$ of Eq.(1) of the form

$$y_k^{(m)}(x,\rho) = ((-1)^k \omega\rho)^m exp((-1)^k(\omega\rho x - i\frac{Q(x)}{\omega}))[1], \quad m = 0,1. \quad (9)$$

Using the Birkhoff-type fundamental systems of solutions, one has

$$\varphi^{(m)}(x,\rho) = A_1(\rho)y_1^{(m)}(x,\rho) + A_2(\rho)y_2^{(m)}(x,\rho), \quad x \in [0,a], \quad (10)$$

$$\varphi^{(m)}(x,\rho) = B_1(\rho)Y_1^{(m)}(x,\rho) + B_2(\rho)Y_2^{(m)}(x,\rho), \qquad x \ge a. \tag{11}$$

Taking (9) and the initial conditions $\varphi(0,\rho) = 1$ and $\varphi'(x,\rho) = -(\beta_1\rho + \beta_0)$ into account, we calculate

$$A_1(\rho) = \frac{\omega + \beta_1}{2\omega}[1], \quad A_2(\rho) = \frac{\omega - \beta_1}{2\omega}[1]. \tag{12}$$

Substituting (9) and (12) in (10), we obtain asymptotic formula for $\varphi^{(m)}(a\text{-}0,\rho)$ $m=1$ as $|\rho|\to\infty$ uniformly in $x\epsilon[0, a]$ Now using (8), (11) and the smooth condition $\varphi^{(m)}(a-0,\rho) = \varphi^{(m)}(a+0,\rho),\ m = 0,1,$ we have

$$B_1(\rho) = \frac{1}{2}exp(-i\rho a(a-1)^{\frac{1}{2}} + Q(a)(a-1)^{\frac{-1}{2}})(i\rho(a-1)^{\frac{1}{2}}\varphi(a,\rho) + \varphi'(a,\rho))[1],$$

$$B_1(\rho) = \frac{1}{2}exp(i\rho a(a-1)^{\frac{1}{2}} - Q(a)(a-1)^{\frac{-1}{2}})(-i\rho(a-1)^{\frac{1}{2}}\varphi(a,\rho) + \varphi'(a,\rho))[1].$$

Substituting these expressions $B_1(\rho)$, $B_1(\rho)$ and (8) in (11), we obtain asymptotic form for $\varphi^{(m)}(x,\rho)$ as $x \ge a$ Lemma 2 is proved. ♦

Denote $$\Delta(\rho) := U(e(x,\rho)). \tag{13}$$

The function $\Delta(\rho)$ is called the characteristic function for the boundary value problem L. The function $\Delta(\rho)$ is holomorphic in Π_+ and Π_- and for real ρ there exist the finite limits

$$\Delta\pm(\rho) = \lim_{z\to\rho,\, z\in\Pi_\pm} \Delta(z).$$

Moreover, the function $|\rho|\to\infty, \rho\in\overline{\Pi}_\pm$ is continuously differentiable for $\rho\in\overline{\Pi}_\pm\setminus\{0\}$

Theorem 3: For $|\rho|\to\infty, \rho\in\overline{\Pi}_\pm$, the following asymptotical formula holds:

$$\Delta(\rho) = \frac{\rho}{2}(a-1)^{\frac{-1}{2}} exp(\pm(i\rho a - Q(a)))((-\omega \pm i(a-1))(1-\frac{\beta_1}{\omega})exp(\omega\rho a - \frac{iQ(a)}{\omega})[1]$$
$$-(-\omega \mp i(a-1))(1+\frac{\beta_1}{\omega})exp(-\omega\rho a + \frac{iQ(a)}{\omega})[1]).$$

Proof: By the Birkhoff-type fundamental system of solutions of Eq.(1) on the interval $[0, a]$ we have

$$e^{(m)}(x,\rho) = A_1(\rho)y_1^{(m)}(x,\rho) + A_2(\rho)y_2^{(m)}(x,\rho), \quad x\in[0,a]. \tag{14}$$

Using of the Cramer's rule, we calculate

$$A_1(\rho) = \frac{1}{2}(a-1)^{\frac{-1}{2}}(1 \mp \frac{i(a-1)}{\omega})exp(\omega\rho a - iQ(a)/\omega)exp(\pm(i\rho a - Q(a)))[1],$$

$$A_2(\rho) = \frac{1}{2}(a-1)^{\frac{-1}{2}}(1 \pm \frac{i(a-1)}{\omega})exp(-\omega\rho a + iQ(a)/\omega)exp(\pm(i\rho a - Q(a)))[1]$$

Now, taking (9), (13) and coefficients $A_j(\rho)$, $j = 1,2$ we have

$$e^{(m)}(x,\rho) = \frac{(\omega\rho)^m}{2}(a-1)^{\frac{-1}{2}} exp(\pm(i\rho a - Q(a)))((-1)^m(1 \mp \frac{i(a-1)}{\omega})exp(\omega\rho a - iQ(a)/\omega)$$
$$\times exp(-\omega\rho x + iQ(x)/\omega)[1] + (1 \pm \frac{i(a-1)}{\omega})exp(-\omega\rho a + iQ(a)/\omega)exp(\omega\rho x - iQ(x)/\omega)[1]).$$

Together with (3) and (13) this yields the characteristic function. ♦

Theorem 4: For sufficiently large *k*, the function $\Delta(\rho)$ has simple zeros of the form:

$$\rho_k = \frac{1}{\omega a}(k\pi i + \frac{iQ(a)}{\omega} + \kappa_1 \pm \kappa_2) + O(k^{-1}), \quad (15)$$

where

$$\kappa_1 = \frac{1}{2} ln \frac{\omega + \beta_1}{\omega - \beta_1}, \qquad \kappa_2 = \frac{1}{2} ln \frac{(-\omega - i(a-1))}{(-\omega + i(a-1))}.$$

Proof: Using characteristic function and Rouche's theorem [7], we obtain a countable set of zeros of $\Delta(\rho)$ of the form (16).

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