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Identities of Vassilev and Jha for Bernoulli numbers

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Abstract: We consider the identities for Bernoulli numbers deduced by Jha and Vassilev.

Key words: Bernoulli and Stirling numbers

INTRODUCTION

Vassilev [1, 2] proved the relation:

$$(-1)^m \left[\binom{m}{0} B_n + \cdots + \binom{m}{m-1} B_{n+m-1}\right] = (-1)^n \left[\binom{n}{0} B_m + \cdots + \binom{n}{n-1} B_{m+n-1}\right], \quad m, n \geq 1, \tag{1}$$

Which is equivalent to:

$$(-1)^m \sum_{j=0}^{M} \binom{m}{j} B_{n+j} = (-1)^n \sum_{k=0}^{N} \binom{n}{k} B_{m+k}, \qquad M = m-1, \quad N = n-1, \tag{2}$$

Involving Bernoulli numbers [3-5]. It is immediate to observe that (2) also is correct if we employ $M = m$ & $N = n$, that is, (2) is the Carlitz's identity [6, 7].

The expression (2) can be written in the form [1]:

$$\sum_{j=m}^{n} \binom{n+1}{j}\binom{j}{m} B_j = (-1)^{n+1} \sum_{k=n+1-m}^{n} \binom{n+1}{k}\binom{k}{n+1-m} B_k, \qquad 1 \leq m \leq n. \tag{3}$$

Jha [8] obtained the property:

$$\sum_{k=0}^{n} \frac{(-1)^{k-1}\, k!}{(k+1)\,(k+2)} S_n^{[k]} = B_{n+1}, \qquad n \geq 0, \tag{4}$$

With the participation of Stirling numbers of the second kind [9-11]. Now we shall realize an elementary proof of (4), in fact, we know the following result [9, 12-15] involving Stirling numbers of the first kind:

$$\sum_{k=0}^{n} B_k\, S_n^{(k)} = \frac{(-1)^n\, n!}{n+1}, \qquad n \geq 0, \tag{5}$$

then:

$$\sum_{k=0}^{n} B_{k+1}\, S_n^{(k)} = \sum_{k=0}^{n} B_{k+1}\left[S_{n+1}^{(k+1)} + n\, S_n^{(k+1)}\right] = \sum_{r=0}^{n+1} B_r\, S_{n+1}^{(r)} + n \sum_{j=0}^{n} B_j\, S_n^{(j)},$$

$$\overset{(5)}{=} \frac{(-1)^{n+1}\,(n+1)!}{n+2} + n\, \frac{(-1)^n\, n!}{n+1},$$

therefore:

$$\sum_{k=0}^{n} B_{k+1}\, S_n^{(k)} = \frac{(-1)^{n-1}\, n!}{(n+1)\,(n+2)}, \qquad n \geq 0. \tag{6}$$

We have the inversion formula [5, 9]:

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$$\sum_{k=0}^{n} f(k)\ S_n^{(k)} = g(n) \qquad \therefore \qquad \sum_{k=0}^{n} g(k)\ S_n^{[k]} = f(k), \tag{7}$$

Whose application in (6) implies (4), q.e.d.

In [16] is the identity:

$$n \sum_{k=1}^{n} \frac{1}{k} B_k\, S_{n-1}^{(k-1)} = -\sum_{k=1}^{n} \frac{1}{k+1} S_n^{(k)}\,, \qquad n \geq 1, \tag{8}$$

However, it was obtained by Shirai-Sato [17].

Jha [18] proved the relation:

$$\sum_{k=1}^{n} \frac{(-1)^k\,(k-1)!}{k+1} S_n^{[k]} = (-1)^{n-1} B_n\,, \qquad n \geq 1, \tag{9}$$

And we can give an alternative proof of (9), in fact, we shall consider the following expression:

$$Q \equiv \sum_{k=0}^{n} (-1)^{k-1} B_k\, S_n^{(k)} \overset{(5)}{=} \sum_{r=0}^{n} \frac{(-1)^{r-1}\,r!}{r+1} \sum_{k=r}^{n} (-1)^k S_k^{[r]}\, S_n^{(k)} = (-1)^{n-1} \sum_{r=0}^{n} \frac{(-1)\quad r!}{r+1} L_n^{[r]}, \tag{10}$$

with the Lah numbers [5, 19]:

$$L_n^{[r]} \equiv \sum_{k=r}^{n} (-1)^{n-k}\ S_k^{[r]}\, S_n^{(k)} = \frac{n!}{r!}\binom{n-1}{r-1}, \tag{11}$$

Then from (10):

$$Q = (-1)^{n-1}\, n!\,(n-1)! \sum_{r=0}^{n} \frac{(-1)^r}{(n-r)!\,(r-1)!\,(r+1)} = \frac{(-1)^n\, n!}{2}\ \ {}_2F_1(1-n, 2; 3; 1),$$

Therefore:

$$\sum_{k=0}^{n} (-1)^{k-1}\ B_k\, S_n^{(k)} = \frac{(-1)^n\,(n-1)!}{n+1}\,, \qquad n \geq 1, \tag{12}$$

where we can apply (7) to deduce (9), q.e.d.

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