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Harmonic and Stirling Numbers

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Abstract: We write the harmonic numbers in terms of Stirling numbers of the second kind.

Key words: Stirling - Harmonic and Generalized Bernoulli numbers

INTRODUCTION

We know that the Stirling numbers of the first kind allow determine the harmonic numbers [1, 2]:

$$H_n \equiv 1 + \frac{1}{2} + \ldots + \frac{1}{n} = \frac{(-1)^n}{n!} \sum_{k=1}^{n} (-1)^k \, k \, S_n^{(k)}, \qquad n \geq 1, \tag{1}$$

But we have the identity [3]:

$$H_n = \frac{(-1)^{n+1}}{n!} S_{n+1}^{(2)}, \tag{2}$$

and the Schläfli's expression [1, 4]:

$$S_n^{(n-k)} = (-1)^k \sum_{j=0}^{k} \binom{k-n}{k+j} \binom{k+n}{k-j} S_{k+j}^{[j]}, \tag{3}$$

Then we can obtain a formula for H_n in terms of $S_j^{[k]}$, that is, to construct the harmonic numbers via the Stirling numbers of the second kind. In fact, from (2) and (3):

$$H_n = \frac{1}{n!} \sum_{k=0}^{n-1} \binom{-2}{n+k-1} \binom{2n}{n-k-1} S_{n+k-1}^{[k]}, \tag{4}$$

and the property:

$$\binom{-m}{j} = (-1)^j \binom{m+j-1}{j}, \tag{5}$$

In particular $\binom{-2}{j} = (-1)^j \, (j+1)$, therefore:

$$\binom{-2}{n+k-1} = (-1)^{n+k+1} \, (n+k), \tag{6}$$

Hence (4) implies the relation:

$$H_{n+1} = \frac{1}{(n+1)!} \sum_{k=0}^{n} (-1)^k \binom{2n+2}{n+k+2} (n+k+1) \, S_{n+k}^{[k]}, \quad n \geq 0. \tag{7}$$

Now we shall indicate an alternative deduction of (7); Srivastava-Todorov [5] showed the formula:

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$$B_n^{(\alpha)} = \sum_{k=0}^{n} \frac{(-1)^k \binom{n+\alpha}{n-k}\binom{k+\alpha-1}{k}}{\binom{n+k}{n}} S_{n+k}^{[k]} , \tag{8}$$

For the generalized Bernoulli numbers [6-8], which for $\alpha = n + 2$ and the expression [7, 9]:

$$B_n^{(n+2)} = (-1)^n \, n! \; H_{n+1} , \tag{9}$$

Imply the identity (7), q.e.d.

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