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Generalized Bernoulli Polynomials

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Abstract: We deduce several connections between the generalized polynomials and numbers of Bernoulli and the Stirling numbers.

Key words: Pochhammer symbol - Generalized Bernoulli polynomials - Stirling numbers

INTRODUCTION

We know the following identity of Srivastava-Todorov [1, 2] for the generalized Bernoulli numbers [3]:

$$B_n^{(z)} = \sum_{k=0}^{n} \frac{(-1)^k \binom{n+z}{n-k}\binom{k+z-1}{k}}{\binom{n+k}{n}} S_{n+k}^{[k]}, \quad (1)$$

Involving the Stirling numbers of the second kind [4], where we can apply $\frac{d}{dz}$ and after to evaluate in $z = 1$ to obtain the expression:

$$\left[\frac{d}{dz} B_n^{(z)}\right]_{z=1} = B_n H_{n+1} - \sum_{k=0}^{n} \frac{(-1)^k}{k+1} \frac{\binom{n+1}{n-k}}{\binom{n+k}{n}} S_{n+k}^{[k]}, \quad (2)$$

With the participation of the harmonics and Bernoulli numbers [4-6]:

$$B_n = B_n^{(1)} = \sum_{k=0}^{n} \frac{(-1)^k \binom{n+1}{n-k}}{\binom{n+k}{n}} S_{n+k}^{[k]}; \quad (3)$$

In the deduction of (2) we use (3) and the relations [5]:

$$\left[\frac{d}{du}\binom{u}{n-k}\right]_{u=n+1} = \binom{n+1}{n-k}(H_{n+1} - H_{k+1}), \qquad \left[\frac{d}{du}\binom{u}{k}\right]_{u=k} = H_k. \quad (4)$$

The property (3) was deduced by Jordan [5, 7-11] and it is immediate from (1). The result (2) is an alternative to the Coffey's formula [2]:

$$\left[\frac{d}{dz} B_n^{(z)}\right]_{z=1} = -\frac{n}{2} B_{n-1} - \sum_{j=0}^{n-2} \binom{n}{j} \frac{B_{n-j} B_j}{n-j}. \quad (5)$$

In [2] we find the property:

$$B_n^{(n)} + n\, B_{n-1}^{(n-1)} = \sum_{k=1}^{n} \frac{1}{k+1} S_n^{(k)}, \quad n \geq 1, \quad (6)$$

Involving the Stirling numbers of the first kind [4-6]. On the other hand, Howard [2, 3, 12] showed the expression:

$$B_n^{(n)} = \sum_{k=0}^{n} \frac{(-1)^k}{k+1} S_n^{(k)}, \quad (7)$$

Then (6) and (7) imply the identity:

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$$\sum_{k=1}^{n} \frac{1}{k+1} S_n^{(k)} = n\,\delta_{1,n} + \sum_{k=1}^{n} \frac{(-1)^k}{k+1} \left[S_{n-1}^{(k)} + S_{n-1}^{(k-1)}\right], \quad n \geq 1. \tag{8}$$

We have the Halphen's formula [5, 13, 14]:

$$\frac{d^n}{dx^n}\left[f(x)\,\phi\left(\frac{1}{x}\right)\right] = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x^{-k} \left[\frac{d^k}{dt^k}\,\phi(t)\right]_{t=\frac{1}{x}} \cdot \frac{d^{n-k}}{dx^{n-k}} \left[x^{-k} f(x)\right], \tag{9}$$

Then we can employ $f(t) = 1, \phi(t) = (t)_n$ to obtain an interesting expression for the *n*th derivative of the Pochhammer symbol:

$$\frac{d^n}{dx^n}\left(\frac{1}{x}\right)_n = n!\; x^{-n} \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n-1}{k-1} x^{-k} B_{n-k}^{(n+1)}\left(1-\frac{1}{x}\right), \quad n \geq 0, \tag{10}$$

Where were applied the relations [2]:

$$\frac{d^k}{dt^k}(t)_n = \frac{(-1)^{n-k}\, n!}{(n-k)!} B_{n-k}^{(n+1)}(1-t)\,, \qquad \frac{d^{n-k}}{dx^{n-k}}\, x^{-k} = \frac{(-1)^{n-k}\,(n-1)!}{(k-1)!}\, x^{-n}\,. \tag{11}$$

If we use (10) for $x = 1$:

$$\left[\frac{d^n}{dx^n}\left(\frac{1}{x}\right)_n\right]_{x=1} = n! \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n-1}{k-1} B_{n-k}^{(n+1)} = \sum_{k=0}^{n} (-1)^k\; k \binom{n}{k} S_{n+1}^{(k+1)}\,, \tag{12}$$

where we employed the formula [5, 15]:

$$S_{n+1}^{(k+1)} = \binom{n}{k} B_{n-k}^{(n+1)}\,. \tag{13}$$

On the other hand, it is very known the relation [4, 5]:

$$\left(\frac{1}{x}\right)_n = \sum_{k=0}^{n} (-1)^{n-k}\; S_n^{(k)}\, x^{-k}\,, \tag{14}$$

Hence:

$$\frac{d^n}{dx^n}\left(\frac{1}{x}\right)_n = x^{-n} \sum_{k=0}^{n} \frac{(-1)^k\,(n+k-1)!}{(k-1)!}\; S_n^{(k)}\, x^{-k}\,, \tag{15}$$

Therefore:

$$\left[\frac{d^n}{dx^n}\left(\frac{1}{x}\right)_n\right]_{x=1} = n! \sum_{k=0}^{n} (-1)^k \binom{n+k-1}{n} S_n^{(k)}\,, \tag{16}$$

whose comparison with (12) gives the identity:

$$n! \sum_{k=0}^{n} (-1)^k \binom{n+k-1}{n} S_n^{(k)} = \sum_{k=0}^{n} (-1)^k\; k \binom{n}{k} S_{n+1}^{(k+1)}\,. \tag{17}$$

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