African Journal of Basic & Applied Sciences 12 (2): 27-28, 2020
ISSN 2079-2034

DOI: 10.5829/idosi.ajbas.2020.27.28

On a Sun's Identity

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Abstract: We give an elementary deduction of the Sun's identity.

Key words: Stirling numbers · Binomial transformation · Harmonic numbers

INTRODUCTION

Here we consider the quantities:

$$Q_n = \sum_{k=0}^{n} \frac{a_{n-k}}{k+1}, \quad n \geq 1, \tag{1}$$

For the sequence [1, 2]:

$$a_n = (-1)^n \int_0^1 \binom{x}{n} dx, \tag{2}$$

That is:

$$a_0 = 1, \ a_1 = -\frac{1}{2}, \ a_2 = -\frac{1}{12}, \ a_3 = -\frac{1}{24}, \ a_4 = -\frac{19}{720}, \ a_5 = -\frac{3}{160}, \ldots \tag{3}$$

From (1) for several values of n:

$$Q_1 = a_1 + \frac{1}{2} a_0, \quad Q_2 + a_1 Q_1 = 0, \quad Q_3 + a_1 Q_2 + a_2 Q_1 = 0,$$
$$Q_4 + a_1 Q_3 + a_2 Q_2 + a_3 Q_1 = 0, \quad Q_5 + a_1 Q_4 + a_2 Q_3 + a_3 Q_2 + a_4 Q_1 = 0, \ldots \tag{4}$$

But (3) implies $Q_1 = 0$, then (4) gives $Q_2 = Q_3 = \ldots = 0$, hence $Q_n = 0$, $n \geq 1$, which is the identity obtained by Zhi-Wei Sun [1]:

$$\sum_{k=0}^{n} \frac{a_{n-k}}{k+1} = 0, \quad n \geq 1. \tag{5}$$

The system (4) means $\sum_{k=0}^{n} a_{n-k} \ Q_k = 0,$ where we can apply (1) and (2) to deduce:

$$a_n = \sum_{r=0}^{n} \frac{(-1)^{n-r}}{r+1} \int_0^1 dx \int_0^1 dy \ \binom{x+y}{n-r}, \tag{6}$$

Because [3]:

$$\sum_{k=r}^{n} \binom{x}{n-k} \binom{y}{k-r} = \binom{x+y}{n-r}. \tag{7}$$

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It is possible to show the property:

$$\int_0^1 dy \sum_{k=0}^{n} \frac{(-1)^k}{k+1} \binom{x+y}{n-k} = \binom{x}{n}, \quad n \geq 0, \tag{8}$$

Then (6) reproduces (2). For $x = 0$, $n \geq 1$, the identity (8) gives (5); with $x = -1$ the relation (8) implies:

$$\sum_{k=0}^{n} \frac{b_{n-k}}{k+1} = 1, \qquad b_n \equiv (-1)^n \int_{-1}^{0} \binom{x}{n} dx, \quad n \geq 0. \tag{9}$$

The expression (8) can be written in the form:

$$\binom{x}{n} = \frac{(-1)^n}{n+1} + \int_x^{x+1} d\eta \sum_{k=0}^{n-1} \frac{(-1)^k}{k+1} \binom{\eta}{n-k}, \quad n \geq 1, \tag{10}$$

Then it is immediate to obtain the known results [4, 5]:

$$[\frac{d}{dx}\binom{x}{n}]_{x=-1} = (-1)^{n+1} H_n, \quad [\frac{d}{dx}\binom{x}{n}]_{x=0} = \frac{(-1)^{n+1}}{n}, \quad [\frac{d}{dx}\binom{x}{n}]_{x=n} = H_n, \quad [\frac{d}{dx}\binom{x}{n}]_{x=1} = \begin{cases} 1, & n=1, \\ \frac{(-1)^n}{n(n-1)}, & n \geq 2, \end{cases} \tag{11}$$

involving harmonic numbers [4-6].

The binomial coefficients have relationship with Stirling numbers of the first kind [4, 5, 7, 8]:

$$\binom{x}{n} = \frac{1}{k!} \sum_{m=0}^{k} S_x^{(m)} x^m, \tag{12}$$

Then (8) gives the following identity:

$$S_n^{(m)} = \frac{}{n+1} \sum_{j=m}^{n} \binom{j}{m} \frac{1}{j+1-m} \sum_{q=0}^{n} (-1)^q \; q! \binom{n+1}{n-q} S_{n-q}^{(j)}, \quad n \geq m \geq 0. \tag{13}$$

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