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On the n^{th} Derivative of Exp (1/x)

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Abstract: We use the Halphen's formula to determine the *n*th derivative of exp(1/x) in terms of the Lah's numbers.

Key words: Stirling numbers - Halphen's formula - Lah's numbers

INTRODUCTION

In [1] the following relation was deduced [2, 3]:

$$\frac{d^n}{dx^n}\, e^{1/x} = (-1)^n\, e^{1/x}\, x^{-n} \sum_{k=0}^{n} L_n^{[k]}\, x^{-k}, \tag{1}$$

with the participation of the Lah's numbers [4, 5]:

$$L_n^{[k]} = \frac{n!}{k!}\binom{n-1}{k-1} = \frac{(n-1)!}{(k-1)!}\binom{n}{k} = (n-k)!\binom{n}{k}\binom{n-1}{k-1}. \tag{2}$$

On the other hand, we have the Halphen's formula [6, 7]:

$$\frac{d^n}{dx^n}\left[f(x)\,\phi\left(\frac{1}{x}\right)\right] = \sum_{k=0}^{n}(-1)^k\binom{n}{k}x^{-k}\,\left[\frac{d^k}{dt^k}\,\phi(t)\right]_{t=\frac{1}{x}} \cdot \frac{d^{n-k}}{dx^{n-k}}\,[x^{-k}\,f(x)], \tag{3}$$

Where we can employ $f(x) = 1$ and $\phi(t) = e^t$, therefore:

$$\frac{d^n}{dx^n}\, e^{1/x} = e^{1/x}\sum_{k=0}^{n}(-1)^k\binom{n}{k}x^{-k}\,\frac{d^{n-k}}{dx^{n-k}}\,x^{-k} = (-1)^n\, e^{1/x}\, x^{-n}\sum_{k=0}^{n}\frac{(n-1)!}{(k-1)!}\binom{n}{k}x^{-k},$$

In agreement with (1) and (2), q.e.d.

In [2] we find that Todorov [8]-Charalambides [9, 10] obtained the identity [11]:

$$A \equiv (-1)^m\sum_{j=0}^{m}(-1)^j\binom{m}{j}\binom{zj}{n} = \frac{m!}{n!}\sum_{k=0}^{n} S_n^{(k)}\, S_k^{[m]}\, z^k, \tag{4}$$

involving the Stirling numbers [7]; we can give a short proof of (4), in fact, we know the expression:

$$\binom{zj}{n} = \frac{1}{n!}\sum_{r=0}^{n} S_n^{(r)}\, j^r z^r, \tag{5}$$

Then:

$$A = \frac{(-1)^m}{n!}\sum_{r=0}^{n} S_n^{(r)}\, z^r \sum_{j=0}^{m}(-1)^j\binom{m}{j}j^r = \frac{(-1)^m}{n!}\sum_{r=0}^{n} S_n^{(r)}\, z^r\,(-1)^m m!\, S_r^{[m]},$$

In harmony with (4), q.e.d. It is possible to show that (4) implies the Comtet's formula [2, 3, 12]:

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$$L_n^{[m]} = \sum_{k=0}^{n} (-1)^{n-k} S_n^{(k)} S_k^{[m]} . \tag{6}$$

Boyadzhiev [2] indicates the relation:

$$\frac{d^n}{dx^n} e^{1/x} = (-1)^n n! \; e^{1/x} x^{-n} L_n^{(-1)}\left(-\frac{1}{x}\right), \tag{7}$$

In terms of the associated Laguerre polynomials of order -1, and the Schwatt's identity [13]:

$$\frac{d^n}{dx^n} e^{1/x} = (-1)^n n! \; e^{1/x} x^{-n} \sum_{k=1}^{n} \frac{(-1)^k}{k!} x^{-k} \sum_{j=1}^{k} (-1)^j \binom{k}{j} \binom{n+j-1}{n}, \tag{8}$$

As alternatives to (1), hence:

$$L_n^{(-1)}(x) = \frac{1}{n!} \sum_{k=0}^{n} L_n^{[k]} (-x)^k = \sum_{k=0}^{n} \binom{n-1}{k-1} \frac{(-x)^k}{k!} . \tag{9}$$

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