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Nörlund Numbers in Terms of Stirling Numbers

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Abstract: We deduce an identity involving the Stirling numbers of the first kind and the Nörlund numbers.

Key words: Stirling numbers - Bernoulli numbers of order k - Nörlund numbers

INTRODUCTION

The Bernoulli numbers of order k are given by [1-6]:

$$\sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!} = \left[\frac{t}{e^t-1}\right]^k , \tag{1}$$

Which allows define the Nörlund numbers [7-10]:

$$N_n = B_n^{(n)} \quad \therefore \quad N_1 = -\frac{1}{2}, \quad N_2 = \frac{5}{6}, \quad N_3 = -\frac{9}{4}, \quad N_4 = \frac{251}{30}, \ldots \tag{2}$$

On the other hand, we have the following relation of Gould [11] – Carlitz [12]:

$$(-1)^k \binom{z}{k} B_k^{(k-z)} = \sum_{j=1}^{k} \binom{k+j-1}{k} \binom{k-z}{k+j} \binom{k+z}{k-j} B_k^{(k+j)}, \quad k \geq 1, \tag{3}$$

Where we can study the case when $z \to 0$, then:

$$(-1)^k N_k = \sum_{j=1}^{k} \binom{k}{k-j} \binom{k+j-1}{k} B_k^{(k+j)} \cdot \lim_{z \to 0} \frac{\binom{k-z}{k+j}}{\binom{z}{k}}, \tag{4}$$

But it is easy to see that:

$$\lim_{z \to 0} \frac{\binom{k-z}{k+j}}{\binom{z}{k}} = (-1)^{k+j-1} \frac{k}{j\binom{k+j}{k}}, \tag{5}$$

Hence from (4):

$$N_k = \sum_{j=1}^{k} (-1)^{j-1} \frac{k}{k+j} \binom{k}{j} B_k^{(k+j)} . \tag{6}$$

Besides, we know the expression [1, 3, 4, 10]:

$$B_k^{(k+j)} = \frac{1}{\binom{k+j-1}{j-1}} S_{k+j}^{(j)} , \tag{7}$$

Involving the Stirling numbers of the first kind, therefore (6) acquires the final form:

$$N_k = \sum_{j=1}^{k} (-1)^{j-1} \frac{k\binom{k}{j}}{j\binom{k+j}{k}} S_{k+j}^{(j)} , \quad k \geq 1, \tag{8}$$

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which permits reproduce the values (2).

Todorov [3, 13, 14] deduced the relation:

$$B_n^{(z)} = \sum_{k=0}^{n} (-1)^k \frac{\binom{n+z}{n-k}\binom{z+k-1}{k}}{\binom{n+k}{k}} S_{n+k}^{[k]}, \quad (9)$$

With the participation of the Stirling numbers of the second kind [3, 4, 15], where we can use $z = n$ to obtain the Liu's relation [9]:

$$N_n = n \sum_{k=0}^{n} \frac{(-1)^k}{n+k} \binom{2n}{n+k} S_{n+k}^{[k]}. \quad (10)$$

Choi [16] applied the multiple Hurwitz zeta function to obtain the following explicit expression for the generalized Bernoulli polynomials [3, 4, 10]:

$$B_{n+k}^{(n)}(x) = n \binom{n+k}{n} \sum_{j=0}^{n-1} (-1)^j \frac{B_{j+k+1}(x)}{j+k+1} \sum_{l=j}^{n-1} \binom{l}{j} S_n^{(l+1)} x^{l-j}, \quad n \geq 1, \quad (11)$$

Such that $B_r^{(m)}(0) = B_r^{(m)}$, and the ordinary Bernoulli numbers are given by $B_m = B_m^{(1)}$. Therefore, (11) with $x = 0$ implies the property:

$$B_{n+k}^{(n)} = n \binom{n+k}{n} \sum_{r=1}^{n} \frac{(-1)^{r-1}}{k+r} B_{k+r} S_n^{(r)}, \quad (12)$$

where we employ $k = 0$ to obtain the Nörlund numbers:

$$N_n = n \sum_{r=1}^{n} \frac{(-1)^{r-1}}{r} B_r S_n^{(r)}. \quad (13)$$

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