African Journal of Basic & Applied Sciences 12 (2): 20-22, 2020
ISSN 2079-2034

DOI: 10.5829/idosi.ajbas.2020.20.22

On a Combinatorial Identity of Egorychev

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Abstract: We employ the hyper geometric function of Gauss to give an elementary proof of a combinatorial property obtained by Egorychev.

Key words: Petkovsek - Wilf-Zeilberger's method - Gauss hyper geometric function

INTRODUCTION

Egorychev [1] deduced the following identity:

$$A \equiv \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{k!}{n^k} S_{k+m}^{[k]} = n^m \,, \qquad m \geq 0, \quad n \geq 1, \tag{1}$$

involving the Stirling numbers of the second kind [2]. For example, if $n = 1, 2, \ldots$ then (1) implies the known expressions [2]:

$$S_N^{[2]} = 2^{N-1} - 1, \qquad S_N^{[3]} = \frac{1}{2}(3^{N-1} + 1 - 2^N), \ldots \tag{2}$$

Here we shall show (1) via the algorithm explained in [3-9], in fact, we have the relation:

$$S_{k+m}^{[k]} = \frac{(-1)^k}{k!} \sum_{j=1}^{k} (-1)^j \binom{k}{j} j^{k+m} \,, \tag{3}$$

then:

$$A = \sum_{j=1}^{n} (-1)^j \, j^m \, B(j,n), \tag{4}$$

where:

$$B(j,n) \equiv \sum_{k=j}^{n} (-1)^k \binom{n-1}{k-1} \binom{k}{j} \left(\frac{j}{n}\right)^k = \left(-\frac{j}{n}\right)^j \binom{n-1}{j-1} \sum_{q=0}^{\infty} t_q \,,$$

such that:

$$t_q = \frac{(-1)^q \, (n-j)! \, (q+j)}{q! \, (n-q-j)! \, j} \left(\frac{j}{n}\right)^q \qquad \therefore \qquad \frac{t_{q+1}}{t_q} = \frac{(q+j-n)\,(q+j+1)}{(q+j)\,(q+1)} \cdot \frac{j}{n} \,,$$

hence:

$$B(j,n) = \left(-\frac{j}{n}\right)^j \binom{n-1}{j-1} \, {}_2F_1\left(-(n-j),\ j+1;\ j;\ \frac{j}{n}\right), \tag{5}$$

involving a Gauss hypergeometric function which has the value 1 if $j = n$.

If now we consider $1 \leq j < n$, then we can apply the following identity [10]:

$${}_2F_1(-a,\ b+1;\ b;\ z) = \frac{1}{b}\,(1-z)^{a-1}\,[\,b - (a+b)\,z\,], \tag{6}$$

with $a = n - j,\ b = j,\ z = \frac{j}{n}$ to obtain the value 0 if $j \neq n$, therefore:

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$$ {}_2F_1\left(-(n-j),\ j+1;\ j;\ \tfrac{j}{n}\right) = \delta_{jn}\,, \tag{7}$$

thus (5) and (7) imply:

$$ B(j,n) = (-\tfrac{j}{n})^j \binom{n-1}{j-1} \delta_{jn} = (-1)^j\, \delta_{jn}\,, \tag{8}$$

and from (4) and (8):

$$ A = \sum_{j=1}^{n} j^m\ \delta_{jn} = n^m\,, \quad q.e.d.$$

Besides, in [1] is the Shoo's identity [11]:

$$ C \equiv \sum_{k=0}^{n} \binom{n}{k}^2 \binom{m+2n-k}{2n} = \binom{m+n}{m}^2, \tag{9}$$

which can be proved via the process indicated in [3-9]:

$$ C = \binom{m+2n}{2n} \sum_{k=0}^{\infty} r_k\,, \qquad r_k = \frac{m!\,(m+2nn-k)!}{(m+2n)!\,(m-k)!} \binom{n}{k}^2 \qquad \therefore \qquad \frac{r_{k+1}}{r_k} = \frac{(k-n)^2\,(k-m)}{(k-2n-m)\,(k+1)^2}\,,$$

hence:

$$ C = \binom{m+2n}{2n}\ {}_2F_1(-n,-n,-m;\ -2n-m,1;1). \tag{10}$$

From [12] we know the expression:

$$ {}_2F_1(-n,a,b;a+b-d-n+1,d;1) = \frac{(d-a)_n\,(d-b)_n}{(d)_n\,(d-a-b)_n}\,, \tag{11}$$

then for $a=-n,\ b=-m,\ d=1$:

$$ {}_2F_1(-n,-n,-m;\ -2n-m,1;1) = \frac{(2n)!}{(2n+m)!}\left[\frac{(n+m)!}{n!}\right]^2, \tag{12}$$

thus (10) and (12) give us the result $C = \binom{m+n}{m}^2,\ q.e.d.$

This hypergeometric approach allows show the Dixon's identity [1]:

$$ D \equiv \sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^3 = (-1)^n\,\frac{(3n)!}{(n!)^3}\,, \tag{13}$$

in fact:

$$ D = \sum_{k=0}^{\infty} q_k\,, \qquad q_k = (-1)^k \binom{2n}{k}^3 \qquad \therefore \qquad \frac{q_{k+1}}{q_k} = \left[\frac{k-2n}{k+1}\right]^3,$$

thus:

$$ D = {}_3F_2(-2n,-2n,-2n;1,1;1) = \frac{(-1)^n\,(3n)!}{(n!)^3}\,,$$

where it was applied the following relation [13]:

$$ {}_3F_2(a,a,a;1,1;\ 1) = \frac{\Gamma\left(1+\frac{a}{2}\right)\Gamma\left(1-\frac{3a}{2}\right)}{\Gamma(1+a)\,\Gamma(1-a)\,\left[\Gamma\left(1-\frac{a}{2}\right)\right]^2}\,, \tag{14}$$

with $a=-2n$.

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