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Identities of Jordan and Shattuck for Stirling Numbers of the Second Kind

E. Gómez-Gómez and J. López-Bonilla

ESIME-Zacatenco, Instituto Politécnico Nacional,
Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México

Abstract: We generalize the Jordan's identity for Stirling numbers of the second kind, which allows give an elementary proof of the Shattuck's formula for these numbers.

Key words: Jordan's identity - Stirling numbers - Shattuck's formula

INTRODUCTION

We know the Jordan's identity [1-5]:

$$Q(0) \equiv \sum_{j=0}^{n} \binom{n}{j} S_j^{[k]} = S_{n+1}^{[k+1]}, \tag{1}$$

Involving Stirling numbers of the second kind with the recurrence relation [4]:

$$S_{n+1}^{[k]} = S_n^{[k-1]} + k\, S_n^{[k]} . \tag{2}$$

Here we consider the quantities:

$$Q(r) \equiv \sum_{j=0}^{n} \binom{n}{j} S_{r+j}^{[k]} , \tag{3}$$

In fact, the application of (2) allows to obtain the sequence:

$$Q(1) = S_{n+2}^{[k+1]} - S_{n+1}^{[k+1]} , \qquad Q(2) = S_{n+3}^{[k+1]} - 2\, S_{n+2}^{[k+1]} + S_{n+1}^{[k+1]} ,$$

$$Q(3) = S_{n+4}^{[k+1]} - 3S_{n+3}^{[k+1} + 3S_{n+2}^{[k+1]} - S_{n+1}^{[k+1]}, \quad Q(4) = S_{n+5}^{[k+1]} - 4S_{n+4}^{[k+1]} + 6S_{n+3}^{[k+1]} - 4S_{n+2}^{[k+1]} + S_{n+1}^{[k+1]} , \ldots$$

Then it is immediate the generalization of (1):

$$\sum_{j=0}^{n} \binom{n}{j} S_{r+j}^{[k]} = \sum_{l=0}^{r} (-1)^{r-l} \binom{r}{l} S_{n+l+1}^{[k+1]} . \tag{4}$$

Hence from (4) for $m \geq 1$:

$$\sum_{r=0}^{m-1} \sum_{j=0}^{n} \binom{m-1}{r} \binom{n}{j} S_{r+j}^{[k]} = \sum_{l=0}^{m-1} (-1)^l S_{n+l+1}^{[k+1]} \sum_{r=l}^{m-1} (-1)^r \binom{r}{l} \binom{m-1}{r} = S_{n+m}^{[k+1]}, \tag{5}$$

Where it was applied the property [5-7]:

$$\sum_{r=l}^{m-1} (-1)^r \binom{r}{l} \binom{m-1}{r} = (-1)^{m-1}\, \delta_{l,m-1} . \tag{6}$$

The identity (5) was deduced by Shattuck [8].

Corresponding Author: J. López-Bonilla, ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 5, 1er. Piso, Col. Lindavista CP 07738, CDMX, México; jlopezb@ipn.mx

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