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On the Namias and Tuenter Identities for Bernoulli Numbers

[1]J. Morales, [1]G. Ovando and [2]J. López-Bonilla

[1]CB1-Área de Física, AMA, Universidad Autónoma Metropolitana-Azcapotzalco,
Av. San Pablo 180, Col. Reynosa-Tamaulipas CP 02200, CDMX, México,
[2]ESIME-Zacatenco, Instituto Politécnico Nacional,
Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México

Abstract: We exhibit an elementary deduction of the Tuenter and Namias relations involving Bernoulli numbers.

Key words: Namias identities - Bernoulli numbers - Tuenter's formulae

INTRODUCTION

Here we consider the quantities:

$$Q(a,b) \equiv \sum_{r=0}^{m-1} \binom{m}{r} a^r B_r\, b^{m-r} = (a\,B+b)^m - a^m\,B_m\,, \tag{1}$$

where it was applied the Rainville's notation $B^k \equiv B_k$ for the Bernoulli numbers [1-8], then:

$$a^{-m}\,Q + B_m = (B+1+c)^m \;= \sum_{r=0}^{m} \binom{m}{r} (B+1)^r\, c^{m-r}\,,\quad c = \frac{b-a}{a}\,,$$

$$= c^m \left[\sum_{r=2}^{m} \binom{m}{r} B_r\, c^{-r} + 1 + \frac{m}{2\,c} \right] = m\,c^{m-1} + \sum_{r=0}^{m} \binom{m}{r} B_r\, c^{m-r}\,, \tag{2}$$

Because [9] $(B+1)^r = B_r\,, r \geq 2$, therefore:

$$B_m + a^{-m}\,Q(a,b) = m\,c^{m-1} + B_m(c), \tag{3}$$

Involving the Bernoulli polynomials [1-5, 9].

We may select the values $a = 2, b = 1 \;\therefore\; c = -\frac{1}{2}$, hence from (2):

$$(-1)^m\,[\,2^m\,B_m + Q(2,1)] = -2\,m + \sum_{k=0}^{m} (-1)^k \binom{m}{k} 2^k\,B_k\,, \tag{4}$$

But we know the Namias identity [10-14]:

$$Q(2,1) \equiv \sum_{r=0}^{m-1} \binom{m}{r} 2^r\,B_r = 2\,(1-2^m)\,B_m\,, \tag{5}$$

Then (4) implies the relation:

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} 2^k\,B_k = 2\,m + (-1)^m\,(2-2^m)\,B_m\,,\quad m \geq 0. \tag{6}$$

We can give a simple proof of the property (5), in fact:

$$Q(2,1) = 2^m \sum_{r=0}^{m-1} \binom{m}{r} B_r \cdot (\tfrac{1}{2})^{m-r} = 2^m \left[B_m\left(\tfrac{1}{2}\right) - B_m \right], \tag{7}$$

Corresponding Author: J. López-Bonilla, ESIME-Zacatenco, Instituto Politécnico Nacional,
Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México. E-mail: jlopezb@ipn.mx

Because [1-5, 9]:

$$B_n(x) = \sum_{j=0}^{n} \binom{n}{j} B_j \cdot x^{n-j} , \tag{8}$$

Besides [9, 15, 16]:

$$B_m\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2^{m-1}} - 1\right) B_m\,,\ m \geq 0, \tag{11}$$

Thus the application of (11) into (7) gives (5), q.e.d.

Now we take the values $a = 2, b = 3 \ \therefore\ c = \frac{1}{2}$, then from (2):

$$Q(2,3) \equiv \sum_{r=0}^{m-1} \binom{m}{r} 2^r B_r \cdot 3^{m-r} = 2\,m - 2^m B_m + \sum_{k=0}^{m} \binom{m}{k} 2^k B_k \overset{(5)}{=} 2\,m + 2(1-2^m) B_m\,, \tag{12}$$

In agreement with Tuenter [17].

We can employ (3) for the values $a = 3, b = 1 \ \therefore\ c = -\frac{2}{3}$:

$$Q(3,1) = -3 \cdot 2^{m-1}(-1)^m\, m + 3^m \left[B_m\left(-\tfrac{2}{3}\right) - B_m\right], \tag{13}$$

However [9]:

$$B_m\left(-\tfrac{2}{3}\right) = (-1)^m \left[B_m\left(\tfrac{2}{3}\right) + m\,\left(\tfrac{2}{3}\right)^{m-1}\right],\ B_m\left(\tfrac{2}{3}\right) = (-1)^m B_m\left(\tfrac{1}{3}\right), \tag{14}$$

Hence from (13):

$$Q(3,1) = 3^m \left[B_m\left(\tfrac{1}{3}\right) - B_m\right]; \tag{15}$$

Similarly, if $a = 3, b = 2, c = -\frac{1}{3}$ then (3) implies:

$$Q(3,2) = 3^m \left[(-1)^m B_m\left(\tfrac{1}{3}\right) - B_m\right], \tag{16}$$

Therefore:

$$Q(3,1) + Q(3,2) = 3^m \left[(1 + (-1)^m) B_m\left(\tfrac{1}{3}\right) - 2\,B_m\right], \tag{17}$$

$$= \begin{cases} 3\,, & m = 1, \\ 0\,, & m\ odd \geq 3, \\ 3(1-3^m)\,B_m, & m\ even, \end{cases}$$

Which is equivalent to the Namias identity [10-14]:

$$\sum_{r=0}^{m-1} \binom{m}{r} 3^r (1 + 2^{m-r}) B_r = 3\,(1-3^m)\,B_m\,,\ \forall\, m \geq 1. \tag{18}$$

The expression (3) gives the relations:

$$Q(3,2) = 3^m \left[(-1)^m B_m\left(\tfrac{1}{3}\right) - B_m\right],\ \ Q(3,4) = 3\,m + 3^m \left[B_m\left(\tfrac{1}{3}\right) - B_m\right], \tag{19}$$

That is:

$$Q(3,2) + Q(3,4) = 3\,m + 3^m\left[(1 + (-1)^m)\,B_m\left(\tfrac{1}{3}\right) - 2\,B_m\right], \quad (20)$$

$$= \begin{cases} 6, & m = 1, \\ 3\,m, & m\ odd \geq 3, \\ 3m + 3(1 - 3^m)\,B_m, & m\ even, \end{cases}$$

In harmony with Tuenter [17]:

$$\sum_{r=0}^{m-1} \binom{m}{r} 3^r (2^{m-r} + 4^{m-r})\,B_r = 3\,m + 3\,(1 - 3^m)\,B_m\,,\ m \geq 1. \quad (21)$$

In (17) and (20) was used the formula [9, 15, 16]:

$$B_m\left(\tfrac{1}{3}\right) = \tfrac{1}{2}\left(\tfrac{1}{3^{m-1}} - 1\right),\ m\ even. \quad (22)$$

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