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An Application of the Burchnall's Formula

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Abstract: We employ the Burchnall's expansion to obtain the Cohen's relation in quantum mechanics.

Key words: Cohen's formula · Burchnall's identity · Laplace-Chebyshev-Hermite polynomials

INTRODUCTION

Cohen [1-3] proved the expansion:

$$(\lambda x+p)^n=\sum_{k=0}^{[\frac{n}{2}]}\ \sum_{r=0}^{n-2k}\frac{(-1)^k\,n!}{k!(n-2)!}\binom{n-2k}{r}(\frac{i\hbar}{2})^k\lambda^{n-k-r}x^{n-2k-r}p^r, \tag{1}$$

Such that all x factors are to the left of the p factors and:

$$p=i\hbar\frac{d}{dx},\qquad [x,p]=i\hbar. \tag{2}$$

On the hand, Burchnall [4-6] obtained the relation:

$$(\frac{d}{dy}-2y)^n=\sum_{t=0}^{n}(-1)^t\binom{n}{t}H_t(y)\frac{d^{n-t}}{dy^{n-t}}, \tag{3}$$

Involving the Laplace [7]-Chebyshev [8]-Hermite [9] polynomials $H_m(x)$ [10-12].

Now we shall show that (3) implies (1), in fact, in (3) we realize the change of variable $y=-\sqrt{\frac{\lambda}{2i\hbar}}x$, where λ is an arbitrary parameter, to deduce:

$$(\lambda x+p)^n=\sum_{t=0}^{n}(-1)^t\binom{n}{t}\left(\frac{\lambda i\hbar}{2}\right)^{\frac{t}{2}}H_t\left(-\sqrt{\frac{\lambda}{2i\hbar}}x\right)p^{n-t}, \tag{4}$$

where we can apply the property [10]:

$$H_r(z)=r!\ 2^r\sum_{q=0}^{[\frac{r}{2}]}\frac{(-1)^q}{q!(r-2q)!2^{2q}}z^{r-2q}, \tag{5}$$

To obtain (1):

$$(\lambda z+p)^n=\sum_{q=0}^{[\frac{n}{2}]}\left(-\frac{i\hbar}{2}\right)^q\frac{n!}{q!}\sum_{r=0}^{n-20}\frac{\lambda^{n-q-r}}{r!(n-2q-r)!}x^{n-2q-r}p^r, \tag{6}$$

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q.e.d. In the deduction of (6) we used the following summation interchange formula [13]:

$$\sum_{t=0}^{n}\sum_{q=0}^{[\frac{t}{2}]} a_{j,k} = \sum_{q=0}^{[\frac{n}{2}]}\sum_{t=0}^{n} a_{j,k}. \tag{7}$$

The Hermite polynomials verify the differential equation [10]:

$$H_n^{''} - 2xH_n^{'} + 2nH_n = 0, \quad n = 0,1,2,... \tag{8}$$

that is:

$$H_0 = 1, \quad H_1 - 2x, \quad H_2 = 4x^2 - 2, \quad H_3 = 8x^3 - 12x,... \tag{9}$$

In according with the following annihilation and creation operators [14]:

$$H_n = \left(2x - \frac{d}{dx}\right)H_{n-1} = (2x - \frac{d}{dx})^n H_0, \tag{10}$$
$$\frac{d}{dx}H_n = 2nH_{n-1} \quad \therefore \quad \frac{d^k}{dx^k}H_n = 2^k \frac{n!}{(n-k)!}H_{n-k}.$$

REFERENCES

1. Cohen, L., 1966. Expansion theorem for functions operators, J. Math. Phys., 7(2): 244-245.
2. García-Quijas, P.C. and L.M. Arévalo-Aguilar, 2007. Factorizing the time evolution operator, Phys. Scr., 75(2): 185-194.
3. Álvarez-Ballesteros, S., J. López-Bonilla and H. Torres-Silva, 2017. Cohen's formula applied to Laplace-Chebyshev-Hermite polynomials, Prespacetime J., 8(1): 93-95.
4. Burchnall, J.L., 1941. A note on the polynomials of Hermite, Quart. J. Math, 12(1): 9-11.
5. Ismail, M.E.H., T.H. Koelink and P. Roman, 2018. Generalized Burchnall type identities for orthogonal polynomials and expansions, SIGMA, 14: 072.
6. Ismail, M.E.H. and N. Saad, 2020. The asymptotic iteration method revisited, J. Math. Phys., 61: 033501.
7. Laplace, P.S., 1810. Mémoire sur les integrals définies, et leur application aux probabilities, et spécialement á la recherché du milieu qu'il faut choisir entre les résultats des observations, Mém. Cl. Sci. Math. Phys. Inst. France, 58: 279-347.
8. Chebyshev, P.L., 1859. Sur le développement des fonctions á une soule variable, Bull. Acad. Sci. St. Petersb., 1: 193-200.
9. Ch. Hermite, 1864. Sur un nouveau développement en série de fonctions, Compt. Rend. Acad. Sci. Paris, 58: 93-100 and 266-273.
10. Abramowitz, M. and I.A. Stegun, 1972. Handbook of mathematical functions, Wiley and Sons, New York.
11. Posadas-Durán, G. and J. López-Bonilla, 2016. On the Saha's generating function for the Hermite polynomials, Prespacetime Journal, 7(13): 1805-1806.
12. López-Bonilla, J., R. López-Vázquez and S. Vidal-Beltrán, 2018. Matrices and orthogonal polynomials, World Scientific News, 103: 245-252.
13. Quaintance, J. and H.W. Gould, 2016. Combinatorial identities for Stirling numbers, World Scientific, Singapore.
14. Morales, J., J.J. Peña, M. Sánchez and J. López-Bonilla, Ladder operators for central potential wave functions from the algebraic representation of orthogonal polynomials, Int. J. Quantum Chem., S25: 155-164.