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Nishimura's Identities Involving Bernoulli Numbers

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Abstract: We deduce some interesting identities involving Bernoulli numbers implied by certain relations of Nishimura.

Key words: Faulhaber-Bernoulli numbers - Nishimura's formulae - Yakubovich's identity

INTRODUCTION

Nishimura [1] studied the expression:

$$R(r) \equiv \frac{2}{r+1}(2^{2r+2}-1)\,B_{2r+2}\,,\quad r \geq 0, \tag{1}$$

Involving Bernoulli numbers [2-10], and he obtained the following properties:

$$R(r) = \sum_{q=0}^{r} \binom{2r+1}{2q} R(q), \tag{2}$$

$$\sum_{q=0}^{r} 2^{2q} \binom{2r+1}{2q+1} R(q) = 1, \tag{3}$$

$$R(r+1) = -\frac{1}{2}\sum_{q=0}^{r} \binom{2r+2}{2q} R(q), \tag{4}$$

Here we shall deduce some identities for B_k implied by them.

Relation (2):
We employ (1) into (2) to obtain:

$$2(2^{2n}-1)\,B_{2n} = 2n\sum_{t=1}^{n} \frac{2^{2t}-1}{t}\binom{2n-1}{2t-2} B_{2t} = \sum_{t=1}^{n} \frac{(2^{2t}-1)(2t-1)}{t}\binom{2n}{2t-1} B_{2t}\,,$$

$$= 2\sum_{t=1}^{n} (2^{2t}-1)\binom{2n}{2t-1} B_{2t} - 1\,, \tag{5}$$

where it was applied the Yakubovich's identity [11, 12]:

$$\sum_{t=1}^{n} \frac{2^{2t}-1}{t}\binom{2n}{2t-1} B_{2t} = 1 \;\text{ that is }\; \sum_{t=1}^{n} (2^{2t}-1)\binom{2n+1}{2t} B_{2t} = n + \frac{1}{2}\,; \tag{6}$$

Hence from (5) and (6):

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$$\sum_{t=1}^{n} (2^{2t} - 1) \binom{2n}{2t-1} B_{2t} = \tfrac{1}{2} + (2^{2n} - 1)\, B_{2n}\,, \;\; n \geq 1. \qquad (7)$$

We know the following Euler's identity [8, 13-17]:

$$\sum_{t=1}^{n} 2^{2t} \binom{2n+1}{2t} B_{2t} = 2\,n\,, \qquad (8)$$

Then from (6) and (8):

$$\sum_{t=1}^{n} \binom{2n+1}{2t} B_{2t} = n - \tfrac{1}{2}\,, \;\; n \geq 1\,. \qquad (9)$$

Relation (3):
We use (1) into (3) to deduce:

$$2 = 4 \sum_{t=1}^{r+1} \frac{2^{2t-2}(2^{2t}-1)}{t} \binom{2r+1}{2t-1} B_{2t} = \sum_{t=1}^{r+1} \frac{4^{2t}}{t} \binom{2r+1}{2t-1} B_{2t} - A\,, \qquad (10)$$

Such that:

$$A \equiv \sum_{t=1}^{r+1} \frac{2^{2t}}{t} \binom{2r+1}{2t-1} B_{2t} = \frac{1}{n} \sum_{t=1}^{n} 2^{2t} \binom{2n}{2t} B_{2t} = \frac{1}{n}\, [2n - 1 + 2(1 - 2^{2n-1})\, B_{2n}\,, n = r+1, \qquad (11)$$

where it was used a Namias identity [8, 14-17]; thus from (10) and (11):

$$\sum_{t=1}^{n-1} 4^{2t} \binom{2n}{2t} B_{2t} = 4n - 1 + (2 - 2^{2n} - 4^{2n})\, B_{2n}, \;\; n \geq 1\,. \qquad (12)$$

With the property [8]:

$$\sum_{t=1}^{n} \binom{2n}{2t} B_{2t} = n - 1 + B_{2n}\,, \qquad (13)$$

and the identity of Namias employed in (11) it is possible to prove the expression [16, 17]:

$$\sum_{t=1}^{n-1} (2^{2t} - 2) \binom{2n}{2t} B_{2t} = 1 + 2\,(1 - 2^{2n})\, B_{2n}\,, \;\; n \geq 1\,. \qquad (14)$$

Relation (4):
The application of (1) into (4) gives:

$$2(2n+1)(1 - 2^{2n+2})\, B_{2n+2} = 2(n+1) \sum_{t=1}^{n} (2^{2t} - 1) \binom{2n+1}{2t-1} B_{2t} - \sum_{t=1}^{n} \binom{2n+2}{2t} B_{2t} -$$

$$- \sum_{t=1}^{n} (2^{2t} - 2) \binom{2n+2}{2t} B_{2t}\,,$$

where we can use (13) and (14) to obtain:

$$\sum_{t=1}^{m-1} (2^{2t} - 1) \binom{2m-1}{2t-1} B_{2t} = \tfrac{1}{2} + 2\,(1 - 2^{2m}) B_{2m}, \;\; m \geq 2\,. \qquad (15)$$

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