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A Note on the Lanczos Derivative

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Abstract: We exhibit an interesting result of Liptaj about the Lanczos derivative.

Key words: Lanczos generalized derivative · Orthogonal derivative

INTRODUCTION

Liptaj [1] obtained the following generalized derivative:

$$f^{(n)}(x) = \lim_{s\to 0} \frac{(-1)^n}{s^n} \int_{-1}^{1} \omega^{(n)}(u) f(x+\varepsilon u)\, du, \tag{1}$$

Such that:

$$\int_{-1}^{1} \omega(u)\, du = 1, \quad \omega(k)(\pm 1) \equiv [\frac{d^k \omega}{du^k}]_{u=\pm 1} = 0, \quad k = 0,1,\ldots,n-1. \tag{2}$$

Here we employ the procedure indicated in [2] to motivate the Liptaj's results.

In fact, we have the Cioranescu [3] – (Haslam-Jones) [4] – Lanczos [5] derivative:

$$f'(x) = \lim_{s\to 0} \frac{3}{2s^3} \int_{-s}^{s} t\, f(t+x) dt, \tag{3}$$

which represents differentiation via integration [6-11]. This generalized derivative for higher orders was studied by Rangarajan-Purushothaman [2, 12-14]. Now, we consider the integral:

$$\int_{-s}^{s} Q_n(\frac{t}{s}) f(x+t) dt = \varepsilon \int_{-1}^{1} Q_n(u) f(x+\varepsilon u) du, \tag{4}$$

and the Taylor expansion allows write:

$$f(x+\varepsilon u) = f(x) + \varepsilon f'(x) u + \ldots + \frac{s^n}{n!} f(n)(x) u^n + \frac{s^{n+1}}{(n+1)!} f^{(n+1)}(x) u^{n+1} + \ldots,$$

then from (4):

$$\frac{1}{\varepsilon^{n+1}} \int_{-s}^{s} Q_n(\frac{t}{s}) f(x+t) dt = \sum_{k=0}^{n-1} \frac{s^{k-n}}{k!} f^{(k)}(x) \int_{-1}^{1} u^k Q_n(u) du + \frac{1}{n!} f^{(n)}(x) \int_{-1}^{1} u^n Q_n(u) du + + \sum_{j=n+1}^{\infty} \frac{s^{j-n}}{j!} f^{(j)}(x) \int_{-1}^{1} u^j Q_n(u) du, \tag{5}$$

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which suggesting to select $Q_n(u)$ with the following structure:

$$Q_n(u) = \frac{d^n \omega}{du^n}. \tag{6}$$

Hence, from (6) and integration by parts:

$$\int_{-1}^{1} u^k Q_n(u)\, du = \int_{-1}^{1} u^k \omega^{(n)}(u)\, du = 0 \quad \text{if} \quad \omega^{(k)}(\pm 1) = 0,\ k = 0,1,\ldots,n-1,$$
$$\int_{-1}^{1} u^n Q_n(u)\, du = \int_{-1}^{1} u^k \omega^{(n)}(u)\, du = (-1)^n n! \quad \text{if} \quad \int_{-1}^{1} \omega(u)\, du = 1, \tag{7}$$

in according with the conditions (2) introduced by Liptaj; the application of (7) into (5) implies (1), q.e.d.

For example, for $n = 1$ and $\omega = \frac{3}{4}(1-u^2)$ the expression (1) gives (3). If $\omega = \frac{(2n+1)!!}{2^{n+1} n!}(1-u^2)^n$, then:

$$\omega^{(n)}(u) = \frac{(-1)^n}{2}(2n+1)!! P_n(u), \tag{8}$$

with the participation of the Legendre polynomials [15] and thus (1) allows deduce the formula of Rangarajan-Purushothaman [2, 12-14]:

$$f^{(n)}(x) = \lim_{s \to 0} \frac{(2n+1)!!}{2s^{n+1}} \int_{-s}^{s} P_n\left(\frac{t}{\varepsilon}\right) f(x+t)\, dt, \tag{9}$$

which reproduces (3) for $n = 1$ because $P_1\left(\frac{t}{s}\right) = \frac{t}{s}$.

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