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# On the Khan's Identities for the Gauss Hypergeometric Function

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**Abstract:** Khan used the fractional calculus technique to obtain identities involving the Gauss hypergeometric function; here we realize comments about some of these identities.

**Key words:** Gauss hypergeometric function · Gamma function · Khan's identities

## INTRODUCTION

Khan [1] applied the fractional calculus approach to obtain the following identities involving the Gauss hypergeometric function [2-6]:

$$ {}_2F_1\left(a,b;\frac{a+b-1}{2};\frac{1}{2}\right)=\sqrt{\pi}\left[\frac{\Gamma(\frac{a+b+1}{2})}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}+\frac{\Gamma(\frac{a+b-1}{2})}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)}\right], \tag{1}$$

$$ {}_2F_1\left(a,b;\frac{a+b-3}{2};\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}\Gamma\left(\frac{a+b-3}{2}\right)\left[\frac{(a+b)^2+4(ab-a-b)+3}{2\Gamma\left(\frac{a+1}{2}\right)\Gamma(\frac{b+1}{2})}+\frac{4\sqrt{\pi}(\frac{a+b-1}{2})}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)}\right], \tag{2}$$

$$ {}_2F_1\left(a,b;\frac{a+b+1}{2};\frac{1+z}{2}\right)=\sqrt{\pi}\Gamma\left(\frac{a+b+1}{2}\right)\left[\frac{1}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}\ {}_2F_1\left(\frac{a}{2},\frac{b}{2};\frac{1}{2};z^2\right)+\frac{2}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)}\ {}_2F_1\left(\frac{a+1}{2},\frac{b+1}{2};\frac{1}{2};z^2\right)\right], \tag{3}$$

where $\Gamma(z)$ is the Gamma function [3, 7-11].

On the other hand, in [12] is the formula:

$$ {}_2F_1\left(a,b:\frac{a+b-m}{2};\frac{1}{2}\right)=\frac{2^{b-1}\Gamma(\frac{a+b-m}{2})}{\Gamma(b)}\sum_{k=0}^{m+1}\binom{m+1}{k}\frac{\Gamma(\frac{b+k}{2})}{\Gamma(\frac{a+k-m}{2})},\quad m=-1,0,1,2,... \tag{4}$$

besides, the Legendre duplication formula gives the relation [3]:

$$2^{b-1}\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{b+1}{2}\right)=\sqrt{\pi}\ \Gamma(b). \tag{5}$$

Therefore, (4) and (5) for $m = -1,0,1,3$ imply (1), (2) and the expressions [12]:

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$$ {}_2F_1\left(a,b;\frac{a+b+1}{2};\frac{1}{2}\right)=\frac{\sqrt{\pi}\Gamma(\frac{a+b+1}{2})}{\Gamma\left(\frac{a+1}{2}\right)\Gamma(\frac{b+1}{2})}, \tag{6} $$

$$ {}_2F_1\left(a,b;\frac{a+b}{2};\frac{1}{2}\right)=\sqrt{\pi}\Gamma\left(\frac{a+b}{2}\right)\left[\frac{1}{\Gamma\left(\frac{a+1}{2}\right)\Gamma(\frac{b}{2})}+\frac{1}{\Gamma\left(\frac{a}{2}\right)\Gamma(\frac{b+1}{2})}\right], \tag{7} $$

thus, for example, ${}_2F_1\left(a,a;a;\frac{1}{2}\right)=2^a.$

The Khan's identity (3) must be reviewed because the correct formula is [13]:

$$ {}_2F_1\left(a,b;\frac{a+b+1}{2};\frac{1+z}{2}\right)=\sqrt{\pi}\ \Gamma\left(\frac{a+b+1}{2}\right)\left[\frac{1}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}\ {}_2F_1\left(\frac{a}{2},\frac{b}{2};\frac{1}{2};z^2\right)+\frac{2z}{\Gamma\left(\frac{a}{2}\right)\Gamma(\frac{b}{2})}\ {}_2F_1\left(\frac{a+1}{2},\frac{b+1}{2};\frac{3}{2};z^2\right)\right], \tag{8} $$

## REFERENCES


1. Faisal Khan, M., 2019. Some new hypergeometric transformations via fractional calculus technique, Appl. Maths. & Information Sci., pp: 13.
2. Dutka, J., 1984. The early history of the hypergeometric function, Arch. Hist. Exact Sci., 31(1): 15-34.
3. Wang, Z.X. and D.R. Guo, 1989. Special functions, World Scientific, Singapore.
4. Forrey, R.C., 1997. Computing the hypergeometric function, J. Comput. Phy., 137: 79-100.
5. Barrera-Figueroa, V., I. Guerrero-Moreno, J. López-Bonilla and S. Vidal-Beltrán, 2018. Some applications of hypergeometric functions, Comput. Appl. Math. Sci., 3(2): 23-25.
6. López-Bonilla, J., J. Morales and G. Ovando, 2019. Hypergeometric proofs of some combinatorial identities, African J. Basic & Appl. Sci., 11(2): 27-29.
7. Davis, P.J., 1959. Leonhard Euler's integral: A historical profile of the Gamma function, Amer. Math. Monthly, 66(10): 849-869.
8. Gronau, D., 2003. Why is the Gamma function so as it is? Teaching Maths. & Comput. Sci., 1: 43-53.
9. Srinivasan, G., 2007. The Gamma function: An eclectic tour, Amer. Math. Monthly, 114(4): 297-315.
10. López-Bonilla, J., J. Yaljá Montiel-Pérez and S. Vidal-Beltrán, 2019. Gamma function, Bernoulli numbers and Bell polynomials, African J. Basic & Appl. Sci., 11(2): 25-26.
11. López-Bonilla, J. and S. Vidal-Beltrán, 2019. Bell polynomials and derivatives of gamma function, Studies in Nonlinear Sci., 4(1): 3-4.
12. http://functions.wolfram.com/HypergeometricFunctions/Hypergeometric2F1/03/04/01/.
13. http://functions.wolfram.com/HypergeometricFunctions/Hypergeometric2F1/17/02/10/.