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Kim's Identities Involving Bernoulli and Euler Polynomials

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Abstract: We give elementary proofs of the Kim's identities for Euler and Bernoulli polynomials.

Key words: Bernoulli and Euler polynomials

INTRODUCTION

Kim [1] obtained the following relations for m, $n \geq 1$:

$$\sum_{k=0}^{2-1}(-1)^{k-1}\left[B_{m+1}\left(\frac{k}{2n}+1\right)-B_{m+1}\left(\frac{k}{2n}\right)\right]=(m+1)\sum_{k=0}^{2n-1}(-1)^{k-1}\left(\frac{k}{2n}\right)^{m}, \tag{1}$$

$$\frac{1}{2}(m+1)E_m(x)=(2_n)^m\sum_{k=0}^{2n-1}(-1)^{k-1}B_{m+1}\left(\frac{x+k}{2n}\right), \tag{2}$$

$$\frac{1}{2}(m+1)E_m=(2n)^m\sum_{k=0}^{2n-1}(-1)^{k-1}B_{m+1}\left(\frac{k}{2n}\right), \tag{3}$$

$$E_m(2n)-E_m=2\sum_{k=0}^{2n-1}(-1)^{k-1}k^m, \tag{4}$$

involving the Euler and Bernoulli polynomials and the Euler numbers $E_m \equiv E_m$ [2-4].

The expression (3) is equal to (2) valued in $x = 0$; besides, (4) is consequence from (1), (3) and (2) for $x = 2n$. The identity (1) is trivial because we know the property [2-4]:

$$B_{m+1}(x+1)-B_{m+1}(x)=(m+1)x^m, \tag{5}$$

then (5) with $x=\frac{k}{2n}$ implies (1).

In [5] we can find the relation:

$$\frac{1}{2}(m+1)E_m(2ny)=(2n)^m\sum_{k=0}^{2n-1}(-1)^{k-1}B_{m+1}\left(y+\frac{k}{2n}\right), \tag{6}$$

which gives (2) for $y=\frac{x}{2n}$.

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The Bernoulli polynomials allow to construct the Euler polynomials via the expressions:

$$E_m(x) = \frac{2^{m+1}}{m+1}\left[B_{m+1}\left(\frac{x+1}{2}\right) - B_{m+1}\left(\frac{x}{2}\right)\right] = \frac{2}{m+1}\left[B_{m+1}(x) - 2^{m+1}B_{m+1}\left(\frac{x}{2}\right)\right], \tag{7}$$

Then it is immediate the following property of the Bernoulli numbers:

$$B_{m+1} \equiv B_{m+1}(0) = \frac{m+1}{2(1-2^{m+1})}E_m(0). \tag{8}$$

The relations (1) – (8) can be verified with the polynomials:

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,$$
$$B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}, \quad B_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x, \ldots \tag{9}$$

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$$E_4(x) = x^4 - 2x^3 + x, \quad E_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^2 - \frac{1}{2}, \ldots$$

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