African Journal of Basic & Applied Sciences 11 (2): 57-58, 2019
ISSN 2079-2034

DOI: 10.5829/idosi.ajbas.2019.57.58

Bahsi-Solak Identities for Harmonic Numbers

[1]H.N. Núñez-Yépez, [2]J. López-Bonilla and [2]V.M. Salazar Del Moral

[1]Departamento de Física, UAM-I, Apdo. Postal 55-534,
Iztapalapa CP 09340, CDMX, México
[2]ESIME-Zacatenco, Instituto Politécnico Nacional,
Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México

Abstract: We employ a general relation obtained by Spiess to deduce the Bahsi-Solak identities involving harmonic numbers.

Key words: Harmonic numbers · Bashi-Solak's identities

INTRODUCTION

Bahsi-Solak [1] proved the relations:

$$\sum_{k=1}^{n} H_k^2 = (n+1)H_{n+1}^2 - (2n+3)H_{n+1} + 2(n+1), \tag{1}$$

$$\sum_{k=1}^{n} k\, H_{k+1}^2 = \frac{1}{2}(n^2+n-2)H_{n+1}^2 - \frac{1}{2}(n^2-3n-7)H_{n+1} + \frac{1}{4}(n^2-9n-10), \tag{2}$$

involving the harmonic numbers $H_n \equiv \sum_{k=1}^{n} \frac{1}{k}$ [2-5], with the known properties:

$$\sum_{k=1}^{n} H_k = (n+1)(H_{n+1}-1), \quad \sum_{k=1}^{n} \frac{H_k}{k+1} = \frac{1}{2}(H_{n+1}^2 - H_{n+1}^{(2)}), \quad \sum_{k=1}^{n} kH_k = \frac{n(n+1)}{2}(H_{n+1} - \frac{1}{2}), \tag{3}$$

where $H_{n+1}^{(2)} \equiv \sum_{k=1}^{n+1} \frac{1}{k^2}$.

Spiess [6] obtained the following general expression for $p \geq 0$ and $q \geq 1$:

$$(p+1)\sum_{k=1}^{n} k^p H_k^q = (n+1)^{p+1} H_n^q - \sum_{j=0}^{p-1} \binom{p+1}{j} \sum_{k=1}^{n} k^j H_j^q + \sum_{j=1}^{q} (-1)^j \binom{q}{j} \sum_{k=1}^{n} k^{p+1-j} H_k^{q-j}, \tag{4}$$

which allows to show (1) and (2) if we employ (3). In fact, from (4) with $p = 0$ and $q = 2$:

$$\sum_{k=1}^{n} H_k^2 = (n+1)H_n^2 - 2\sum_{k=1}^{n} H_k + \sum_{k=1}^{n} \frac{1}{k} \overset{(3)}{=} (n+1)H_n^2 - 2(n+1)(H_{n+1}-1) + H_n,$$

Equivalent to (1) because $H_n = H_{n+1} - \frac{1}{n+1}$.

Corresponding Author: J. López-Bonilla, ESIME-Zacatenco, Instituto Politécnico Nacional,
Edif. 4, 1er. Piso, Col. Lindavista CP 07738, CDMX, México.

On the other hand:

$$\sum_{k=1}^{n} kH_{k+1}^2 = \sum_{k=1}^{n} k(H_k + \frac{1}{k+1})^2 = \sum_{k=1}^{n} kH_k^2 + 2\sum_{k=1}^{n} \frac{k}{k+1} H_k + \sum_{k=1}^{n} \frac{k}{(k+1)^2}, \tag{5}$$

But it is simple to see that:

$$\sum_{k=1}^{n} \frac{k}{k+1} H_k = \sum_{k=1}^{n} H_k - \sum_{k=1}^{n} \frac{H_k}{k+1} \overset{(3)}{=} (n+1)(H_{n+1} - 1) - \frac{1}{2}\left(H_{n+1}^2 - H_{n+1}^{(2)} \right),$$

$$\sum_{k=1}^{n} \frac{k}{(k+1)^2} = \sum_{j=1}^{n+1} \frac{j-1}{j^2} = H_{n+1} - H\frac{(2)}{n+1}. \tag{6}$$

Besides, from (4) for $p = 1$ and $q = 2$:

$$\sum_{k=1}^{n} kH_k^2 = \frac{1}{2}(n+1)^2 H_n^2 - \frac{1}{2}\sum_{k=1}^{n} H_k^2 - \sum_{k=1}^{n} k\, H_k + \frac{n}{2},$$

$$\overset{(1),(3)}{=} \frac{1}{2}(n+1)^2 H_n^2 - \frac{1}{2}(n+1)H_{n+1}^2 + \frac{1}{2}(n+3-n^2)H_{n+1} + \frac{1}{4}(n^2 - n - 4), \tag{7}$$

then, (6) and (7) into (5) implies (2) q.e.d.

REFERENCES

1. Bashi, M. and S. Solak, 2010. On the matrices with harmonic numbers, Gazi Univ. J. Sci., 23(4): 445-448.
2. Srivastava, H.M. and J. Choi, 2012. Zeta and q-zeta functions and associated series and integrals, Elsevier, London.
3. Quaintance, J. and H.W. Gould, 2016. Combinatorial identities for Stirling numbers, World Scientific, Singapore.
4. Iturri-Hinojosa, A., J. López-Bonilla and R. López-Vázquez, 2017. Harmonic, Stirling and Bernoulli numbers, Prespacetime J., 8(9): 1173-1175.
5. López-Bonilla, J., R. López-Vázquez and A. Zúñiga-Segundo, 2018. Some properties of the harmonic numbers, Comput. Appl. Math. Sci., 3(2): 21-22.
6. Spiess, J., 1990. Some identities involving harmonic numbers, Maths. of Comput., 55(192): 839-863.