African Journal of Basic & Applied Sciences 11 (2): 52-54, 2019
ISSN 2079-2034

DOI: 10.5829/idosi.ajbas.2019.52.54

On an Identity of Sun for Bernoulli Polynomials

J. López-Bonilla, S. Vidal-Beltrán and A. Zaldívar-Sandoval

ESIME-Zacatenco, Instituto Politécnico Nacional,
Edif. 5, 1er. Piso, Col. Lindavista CP 07738, CDMX, México

Abstract: We employ the Taylor's series to show an identity of Sun involving Bernoulli polynomials, which implies the Carlitz's identity for Bernoulli numbers.

Key words: Sun's formula · Bernoulli polynomials · Carlitz's identity

INTRODUCTION

Sun [1, 2] obtained the following identity involving Bernoulli polynomials [3-7]:

$$A(x) \equiv (-1)^l \sum_{r=0}^{l} \binom{l}{r} x^{l-r} B_{k+r}(z) = (-1)^k \sum_{j=0}^{k} \binom{k}{j} x^{k-j} B_{l+j}(y), \quad x+y+z=1; \quad kl \geq 0, \tag{1}$$

which for $x = 1$, $y = z = 0$ implies the Carlitz's identity [1, 2, 8-13]:

$$(-1)^l \sum_{r=0}^{l} \binom{l}{r} B_{k+r} = (-1)^k \sum_{j=0}^{k} \binom{k}{j} B_{l+j}. \tag{2}$$

Here we give a simple proof of (1), in fact, into $A(x)$ we use $z = 1 - (x + y)$ and the relations [3-7, 14]:

$$B_{k+r}(1-(x+y)) = (-1)^{k+r} B_{k+r}(x+y) = (-1)^{k+r} \sum_{q=0}^{k+r} \binom{k+r}{q} x^{k+r-q} B_q(y), \tag{3}$$

to obtain:

$$A(x) = (-1)^{k+l} \sum_{r=0}^{l} \binom{l}{r} (-1)^r \sum_{q=0}^{k+r} \binom{k+r}{q} x^{k+l-q} B_q(y), \tag{4}$$

where we can apply Taylor's series about $x = 0$ because are immediate the derivatives of (4):

$$A^{(t)}(0) = (-1)^k t! \sum_{m=0}^{t} (-1)^m \binom{k+l-m}{k+l-t} \binom{l}{l-m} B_{k+l-m}(y), \tag{5}$$

therefore:

$$A(x) = \sum_{t=0}^{k+l} \frac{1}{t!} A^{(t)}(0) x^t = (-1)^k \sum_{t=0}^{k+l} [\sum_{m=0}^{t} (-1)^m \binom{k+l-m}{k+l-t} \binom{l}{m}] B_{k+l-t}(y) x^t. \tag{6}$$

Finally, the expression (3.49) of Gould [15] allows show that:

Corresponding Author: J. López-Bonilla, ESIME-Zacatenco, Instituto Politécnico Nacional,
Edif. 5, 1er. Piso, Col. Lindavista CP 07738, CDMX, México.

$$\sum_{m=0}^{t}(-1)^m\binom{k+l-m}{k+l-t}\binom{l}{m}=\binom{k}{t}, \tag{7}$$

whose application into (6) implies the Sun's identity (1), q.e.d.

Pan-Sun [14] obtained the interesting relation:

$$\frac{(-1)^m}{m}\sum_{k=0}^{m}\binom{m}{k}B_{m-k}(x)B_{n-1+k}(y)-\frac{B_m(z)}{m}B_{n-1}(y)= \qquad x+y+z=1,\ m,n\geq 1$$
$$=\frac{(-1)^n}{n}\sum_{r=0}^{n}\binom{n}{r}B_{n-r}(x)B_{m-1+r}(z)-\frac{B_n(y)}{n}B_{m-1}(z), \tag{8}$$

which for $x = 1$, $y = z = 0$ implies the following Woodcock's identity involving Bernoulli numbers [16]:

$$\frac{(-1)^m}{m}\sum_{q=0}^{m-1}\binom{m}{q}(-1)^q B_q B_{m+n-q-1}=\frac{(-1)^n}{n}\sum_{j=0}^{n-1}\binom{n}{j}(-1)^j B_j B_{m+n-j-1}, \tag{9}$$

where we can employ $n = 1$ to deduce the known Euler's result:

$$\frac{1}{m}\sum_{r=1}^{m}\binom{m}{r}B_r B_{m-r}+B_{m-1}=-B_m,\quad m\geq 1. \tag{10}$$

For $m, n \geq 2$ the expression (9) adopts the form:

$$\frac{(-1)^m}{m}\sum_{q=0}^{m-1}\binom{m}{q}B_q B_{m+n-q-1}=\frac{(-1)^n}{n}\sum_{j=0}^{n-1}\binom{n}{j}B_j B_{m+n-j-1}, \tag{11}$$

that for example, for $n = 2$, gives the property:

$$\frac{(-1)^m}{m}\sum_{q=0}^{m-1}\binom{m}{q}B_q B_{m-q+1}=\frac{1}{2}(B_{m+1}-B_m). \tag{12}$$

REFERENCES

1. Sun, Z.W., 2003. Combinatorial identities in dual sequences, European J. Combin., 24(6): 709-718.
2. Chen, W.Y.C. and L.H. Sun, 2009. Extended Zeilberger's algorithm for identities on Bernoulli and Euler Polynomials, J. Number Theory, 129: 2111-2132.
3. Lehmer, D.H., 1988. A new approach to Bernoulli polynomials, Amer. Math. Monthly, 95: 905-911.
4. Temme, N.M., 1996. Special functions. An introduction to the classical functions of Mathematical Physics, John Wiley & Sons, New York.
5. Srivastava, H.M. and J. Choi, 2012. Zeta and q-zeta functions and associated series and integrals, Elsevier, London.
6. Arakawa, T., T. Ibukiyama and M. Kaneko, 2014. Bernoulli numbers and zeta functions, Springer, Japan.
7. Quaintance, J. and H.W. Gould, 2016. Combinatorial identities for Stirling numbers, World Scientific, Singapore.
8. Carlitz, L., 1968. Bernoulli numbers, Fibonacci Quart., 6: 71-85.
9. Wu, K.J., Z.W. Sun and H. Pan, 2004. Some identities for Bernoulli and Euler polynomials, Fibonacci Quart., 42: 295-299.

10. Vassilev, P. and M. Vassilev-Missana, 2008. Some relations involving Bernoulli numbers, Notes on Number Theory and Discrete Maths., 14(1): 10-12.
11. López-Bonilla, J., J. Yaljá Montiel-Pérez and A. Zaldívar-Sandoval, 2017. Comments on some identities of Muthumalai, Vassilev and Shirai-Sato, BAOJ Phys., 2(4): 1-2.
12. Dolgy, D.V., D.S. Kim, J. Kwon and T. Kim, 2019. Some identities of ordinary and degenerate Bernoulli numbers and polynomials, Symmetry, 11: 847.
13. Cruz-Santiago, R., J. López-Bonilla and J. Yaljá Montiel-Pérez, 2019. On some identities for Bernoulli numbers, African J. Basic & Appl. Sci., 11(2): 30-31.
14. Pan, H. and Z.W. Sun, 2006. New identities involving Bernoulli and Euler polynomials, J. Comb. Theory, A113: 156-175.
15. Gould, H.W., 1972. Combinatorial identities, Morgantown, W.Va.
16. Woodcock, C.F., 1979. Convolutions on the ring of p-adic integers, J. London Math. Soc., 20: 101-108.