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On Some Identities for Bernoulli Numbers

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Abstract: We exhibit elementary proofs for the identities involving polynomials and numbers of Bernoulli recently obtained by Dolgy *et al*.

Key words: Bernoulli polynomials · Carlitz's identity · Bernoulli numbers

INTRODUCTION

In [1] were obtained the following identities involving numbers and polynomials of Bernoulli [2-5]:

$$(-1)^m \sum_{r=0}^{m} \binom{m}{r} B_{n+r} = (-1)^n \sum_{j=0}^{n} \binom{n}{j} B_{m+j} \quad m, n = 1, 2, ... \tag{1}$$

$$B_n(2) = n + B_n + \delta_{n,1}, \quad n \geq 0, \tag{2}$$

$$A \equiv \sum_{j=k}^{m} \binom{m}{j} \binom{j}{k} B_{m-j} = \binom{m}{k} B_{m-k}, \quad m-k \geq 2, \quad m, k = 1, 2, ... \tag{3}$$

$$B_{m+1}(n+1) - B_{m+1} = (n+1)^{m+1} - (n+1) - \sum_{j=1}^{m-1} \binom{m+1}{j} \frac{B_{j+1}(n+1) - B_{j+1}}{j+1}, \quad m, n = 1, 2, ... \tag{4}$$

The relation (1) was deduced by Carlitz [6-11]; here we give elementary proofs for (2), (3) and (4). In fact, from [2]:

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k \, x^{n-k}, \quad n \geq 0, \tag{5}$$

that is:

$$B_0(2) = B_0 = 1, \quad B_1(2) = 2 + B_1 = 1 + B_1 + \delta_{1,1}, \tag{6}$$

and from (5) for $n \geq 2$

$$B_n(2) = 2^n \left[\sum_{k=0}^{n-1} \binom{n}{k} \frac{B_k}{2^k} + \frac{B_n}{2^n} \right] = n + B_n, \tag{7}$$

because we use the known expression:

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$$\sum_{k=0}^{n-1}\binom{n}{k}\frac{B_k}{2^k}=\frac{n}{2^n},\quad n\geq 2, \tag{8}$$

thus (6) and (7) imply (2), q.e.d.

We have the property [5]:

$$\sum_{r=0}^{n}\binom{n}{k}B_r=B_n,\quad n\geq 2, \tag{9}$$

then:

$$A=\sum_{j=k}^{m}\frac{m!}{(m-j)!k!(j-k)!}B_{m-j}=\frac{m!}{k!(m-k)!}\sum_{t=0}^{m-k}\binom{m-k}{t}B_t\overset{(9)}{=}\binom{m}{k}B_{m-k}=(3),\ \text{q.e.d.}$$

We know the relation [3-5, 12]:

$$B_{j+1}(n+1)-B_{j+1}=(j+1)\sum_{r=0}^{n}r^j, \tag{10}$$

therefore:

$$\begin{aligned}\sum_{j=1}^{m-1}\binom{m+1}{j}\frac{B_{j+1}(n+1)-B_{j+1}}{j+1}&=\sum_{r=0}^{n}\sum_{j=1}^{m-1}\binom{m+1}{j}r^j,\\ &=\sum_{r=0}^{n}[\sum_{j=0}^{m+1}\binom{m+1}{j}r^j-1-(m+1)r^m-r^{m+1}],\\ &=\sum_{t=1}^{n+1}t^{m+1}-(n+1)-(m+1)\sum_{r=0}^{n}r^m-\sum_{r=0}^{n}r^{m+1},\\ &=(n+1)^{m+1}-(n+1)-(m+1)\sum_{r=0}^{n}r^m,\end{aligned}$$

where we can apply (10) for $j=m$ to deduce (4), q.e.d.

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