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Hypergeometric Proofs of Some Combinatorial Identities

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Abstract: We use hypergeometric functions to show some combinatorial relations.

Key words: Gauss hypergeometric function · Binomial coefficients

INTRODUCTION

The Gauss hypergeometric function [1-3] verifies the identity [4]:

$$ {}_2F_1(-m,\ q;\ c;\ 1) = \frac{(c-q)_m}{(c)_m}, \qquad m \ge 0, \tag{1}$$

which allows prove the relation:

$$ {}_2F_1(k-n,\ k-a;\ k-b;\ 1) = \frac{(b-a)!(b-n)!}{(b-k)!(b-a+k-n)!}, \qquad 0 \le k \le n, \quad b \ne 0,1,\ \dots,\ n-1, \tag{2}$$

employing properties of the gamma function [2, 3] and Pochhammer-Barnes symbols [5-8].

On the other hand, Abel-Arends [9] showed the combinatorial expression:

$$ A \equiv \sum\nolimits_{j=k}^{n} (-1)^{j-k} \binom{n}{j}\binom{j}{k} \frac{\binom{a}{j}}{\binom{b}{j}} = \frac{\binom{a}{k}\binom{b-a}{n-k}}{\binom{b}{n}}, \quad b \ne 0,1,\ \dots,\ n-1, \quad 0 \le k \le n, \tag{3}$$

then here we shall give a hypergeometric proof of (3) using the algorithm of Petkovsek-Wilf-Zeilberger [10-13]. In fact, (3) can be written in the form:

$$ A = \frac{\binom{n}{k}\binom{a}{k}}{\binom{b}{k}} \sum\nolimits_{r=0}^{\infty} t_r, \quad t_r \frac{(-1)^r \binom{n-k}{r}\binom{b-k}{r}}{\binom{b-k}{r}} \qquad \therefore \qquad \frac{t_{r+1}}{t_r} = \frac{(r+k-n)(r+k-a)}{(r+k-b)(r+1)},$$

thus:

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$$A = \frac{\binom{n}{k}\binom{a}{k}}{\binom{b}{k}} \;{}_2F_1(k-n, k-a; k-b; 1) \overset{(2)}{=} \frac{\binom{a}{k}\binom{b-a}{n-k}}{\binom{b}{n}}, \quad \text{q.e.d}$$

In [14] is the following property:

$$ {}_2F_1\left(-n, \frac{1}{2};\ 1;\ 2\right) = \frac{n![1+(-1)^n]}{2^{n+1}[\left(\frac{n}{2}\right)!]^2}, \quad n \geq 0, \tag{4}$$

and from [15]:

$$ {}_2F_1(-n, b; c; z+1) = \frac{(c-b)_n}{(c)_n},\ {}_2F_1(-n, b;\ 1-c+b-n; -z), \tag{5}$$

which for $b = \frac{1}{2},\ c = -n + \frac{1}{2},\ z = -2$ implies the relation:

$$ {}_2F_1\left(-n, \frac{1}{2}; -n + \frac{1}{2}; -n\right) = \frac{(-1)^n n!}{(\frac{1}{2}-n)_n}\ {}_2F_1\left(-n, \frac{1}{2}; 1;\ 2\right) \overset{(5)}{=} \frac{2^{2n-1}(n!)^3}{(2n)!} \cdot \frac{[1+(-1)^n]}{[\left(\frac{n}{2}\right)!]^2}. \tag{6}$$

Spivey [16, 17] obtained the identity:

$$B \equiv \sum_{k=0}^{n} (-1)^k \binom{2k}{k}\binom{2n-2k}{n-k} = \begin{cases} 2n\binom{n}{n/2}, & n\ even \\ 0, & n\ even \end{cases}, \tag{7}$$

and now we give a hypergeometric demonstration of this expression:

$$B = \binom{2n}{n} \sum_{k=0}^{\infty} \quad t_k = \frac{(-1)^k \binom{2k}{k}\binom{2n-2k}{n-k}}{\binom{2n}{n}}, \quad \therefore \quad \frac{t_{r+1}}{t_k} = \frac{(k-n)(k+\frac{1}{2})}{(k-n+\frac{1}{2})(k+1)},$$

hence:

$$B = \binom{2n}{n}\ {}_2F_1\left(-n, \frac{1}{2}; -n + \frac{1}{2}; -1\right) \overset{(6)}{=} eq.(7), \text{ q.e.d.}$$

It is known the relation:

$$ {}_kF_{k-1}(m-n, m-n, \ldots, m-n; -n, -n, \ldots, -n; 1) = 0,\ \ k, m \geq 1,\ k, m \geq 1,\ km < n,\ n \geq 2,\ n-m \geq 1, \tag{8}$$

for example, ${}_2F_1(-2,-2;-3;1) = {}_3F_2(-3,-3,-3;-4,-4;1) = 0,$ etc.

On the other hand, Ridenhour-Grimmett [18, 19] deduced the identity:

$$C \equiv \sum_{j=1}^{n-m} (-1)^j \binom{n}{j}\binom{n-j}{m}^k = 0, \quad n \geq 2, \quad k, m \geq 1, \quad km < n, \tag{9}$$

then:

$$C = \binom{n}{m}^k \sum_{j=0}^{\infty} t_j, \quad t_j = \frac{(-1)^j \binom{n}{j}\binom{n-j}{m}^k}{\binom{n}{m}^k} \qquad \therefore \quad \frac{t_{j+1}}{t_j} = \frac{(j+m-n)^k}{(j-n)^{k-1}(j+1)},$$

thus:

$$C = \binom{n}{m}^k \; {}_kF_{k-1}(m-n, \ldots, m-n; -n, \ldots, -n; 1) \overset{(8)}{=} 0,$$

in according with (9).

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