African Journal of Basic & Applied Sciences 11 (2): 25-26, 2019
ISSN 2079-2034

DOI: 10.5829/idosi.ajbas.2019.25.26

Gamma Function, Bernoulli Numbers and Bell Polynomials

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Abstract: We employ the connection between the Newton's recurrence relation and the complete Bell polynomials to obtain an identity of Euler for the Bernoulli numbers and an expression for the derivatives of gamma function at x = 1 in terms of the Riemann zeta function.

Key words: Gamma function · Bell polynomials · Riemann zeta function · Bernoulli numbers

INTRODUCTION

If the quantities a_k are determined via the Newton's recurrence relation [1]:

$$r a_r + s_1 a_{r-1} + s_2 a_{r-2} + \ldots + s_{r-1} a_1 + s_r = 0, \; a_0 = 1, \; r = 1,2,\ldots, \tag{1}$$

then [2-5]:

$$k! a_k = Y_k(-0! s_1, -1! s_2, -2! s_3, -3! s_4, \ldots, -(k-2)! s_{k-1}, -(k-1)! s_k). \tag{2}$$

with the participation of the complete Bell polynomials [2-13].

In Sec. 2 we apply (1) and (2) to the Gamma function [14] and Bernoulli numbers [6, 7, 11, 14].

Bell Polynomials: We know [15] the following expression for the derivatives of gamma function at $x = 1$:

$$\Gamma^{(k)}(1) = Y_k(-0!\zeta(1), 1!\zeta(2), -2!\zeta(3), \ldots, (-1)^k (k-1)!\zeta(k)), \; k = 1,2,\ldots \tag{3}$$

with the values of Riemann zeta function at 2, 3, 4 … and the notation $\zeta(1) \equiv \gamma$ for the Euler-Mascheroni's constant [14, 16-18]. The comparison of (2) and (3) allows use (1) with $a_r = \frac{\Gamma^{(r)}(1)}{r!}$ and $s_k = (-1)^{k+1}\zeta(r)$, then we obtain the recurrence relation [14]:

$$\Gamma^{(r+1)}(1) = \sum_{k=0}^{r} (-1)^{k+1} k! \binom{r}{k} \Gamma^{(r-k)}(1) \zeta(k+1), \tag{4}$$

that is:

$$\Gamma^{(1)}(1) = -\gamma, \; \Gamma^{(2)}(1) = \gamma^2 + \zeta(2), \; \Gamma^{(3)}(1) = -\gamma^3 - 3\gamma\zeta(2) - 2\zeta(3), \ldots$$

We have [15] a relation for the Bernoulli numbers in terms of the Bell polynomials:

$$B_k = Y_k\left(\frac{B_1}{1}, -\frac{B_2}{2}, \frac{B_3}{3}, \ldots, (-1)^{k+1}\frac{B_k}{k}\right), \tag{5}$$

with the structure (2), hence (1) for $a_r - \frac{B_r}{r!}$ and $s_k = (-1)^k \frac{B_k}{k!}$ implies the following identity:

$$B_{r+1} = r! \sum_{k=0}^{r} (-1)^k \frac{B_{r-k}}{(r-k)!} \frac{B_{k+1}}{(k+1)!}, \tag{6}$$

which gives 0 = 0 if r is an even integer and when r is odd then it can be written in the form of Euler [14, 19-25]:

$$\sum_{k=1}^{n-1} \frac{B_{2n-2k}}{(2n-2k)!} \frac{B_{2k}}{(2k)!} = -\frac{2n+1}{(2n)!} B_{2n}, \; n = 2,3,4,\ldots \tag{7}$$

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