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Coefficients of the Characteristic Polynomial

J. López-Bonilla, R. López-Vázquez and S. Vidal-Beltrán

ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso,
Col. Lindavista CP 07738 CDMX, México

Abstract: We show several representations of the coefficients of the characteristic equation of any matrix A_{nxn}, especially in terms of the (exponential) complete Bell polynomials.

Key words: Complete Bell polynomials · Characteristic polynomial

INTRODUCTION

For an arbitrary matrix $A_{nxn} = (A^i_j)$ its characteristic polynomial [1-3]:

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + ... + a_{n-1}\lambda + a_n, \quad (1)$$

can be obtained, through several procedures [1, 4-7], directly from the condition det $(A^i_j - \lambda\delta^i_j)$ The approach of Leverrier-Takeno [4, 8-13] is a simple and interesting technique to construct (1) based in the traces of the powers $\mathbf{A}^r$, $r = 1, ..., n$. In fact, if we define the quantities:

$$a_0 = 1, \quad s_k = tr\, \mathbf{A}^k, \qquad k = 1,2, ..., n, \quad (2)$$

then the process of Leverrier-Takeno implies (1) wherein the a_i are determined with the following recurrence relation:

$$ra_r + s_1\, a_{r-1} + s_2\, a_{r-2} + ... + s_{r-1}\, sa_1 + s_r = 0, \qquad r = 1,2,...,n, \quad (3)$$

therefore:

$$a_1 = -s_1, \quad 2!\, a_2 = (s_1)^2 - s_2, \quad 3!\, a_3 = -(s_1)^3 + 3s_1s_2 - 2s_3,$$

$$4!\, a_4 = (s_1)^4 - 6(s_1)^2 s_2 + 8s_1\, s_3 + 3(s_2)^2 - 6s_4, \quad (4)$$

$$5!\, a_5 = -(s_1)^5 + 10(s_1)^3 s_2 - 20(s_1)^2 s_3 - 15s_1(s_2)^2 + 30s_1\, s_4 + 20s_2s_3 - 24s_5, \; ...$$

in particular, det $\mathbf{A} = (-1)^n a_n$, that is, the determinant of any matrix only depends on the traces s_r, which means that $\mathbf{A}$ and its transpose have the same determinant.

In [14-16] we find the general expression:

$$m!\, a_m = (-1)^m \begin{vmatrix} s_1 & s_2 & s_3 & \cdots & s_{m-1} & s_m \\ m-1 & s_1 & s_2 & \cdots & s_{m-2} & s_{m-1} \\ 0 & m-2 & s_1 & \cdots & s_{m-3} & s_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & s_1 \end{vmatrix}, \quad m = 1,...,n. \quad (5)$$

which allows reproduce the expressions (4). In Sec. 2 we exhibit several expressions to determine the coefficients a_j, especially via the complete Bell polynomials [17-25].

Characteristic Polynomial: First we consider the polynomial of degree m:

$$f_m(t) = -\left(s_1 t + \frac{1}{2} s_2 t^2 + \frac{1}{3} s_3 t^3 + ... + \frac{1}{m} s_m t^m \right) \quad \therefore$$
$$f_m^{(r)}(0) = -(r-1)!\, s_r, \; 1 \le r \le m, \quad (6)$$

then we obtain the following representation for the coefficients in (1):

$$m!\, a_m = [\frac{d^m}{dt^m} e^{f_m(t)}]_{t=0} = [\frac{d^m}{dt^m} \exp\left(-\sum_{k=1}^{m} \frac{1}{k} s_k t^k \right)](0), \quad m = 1,2,...,n, \quad (7)$$

in harmony with the values (4).

If in (7) we realize the corresponding derivatives, then appears the representation:

Corresponding Author: Dr. J. López-Bonilla, ESIME-Zacatenco, Instituto Politécnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP 07738 CDMX, México.

$$m!\ a_m = \sum \frac{m!}{k_1!k_2!...k_m!}(-\frac{s_1}{1})\ k_1\ (-\frac{s_2}{2})\ k_2\ (-\frac{s_3}{3})\ k_3\ ...\ (-\frac{s_m}{m})\ k_m, \tag{8}$$

where the sum is taken over all partitions such that $k_1 + 2\,k_2 + 3\,k_3 + ... + m\,k_m = m$. On the other hand, the (exponential) complete Bell polynomials are given by [17-26]:

$$Y_m(x_1, x_2,\ ...,x_m) \sum \frac{m!}{k_1!\ ...\ k_m!}(\frac{s_1}{1!})\ k_1\ (\frac{s_2}{2!})\ k_2\ ...\ (-\frac{x_m}{m!})\ k_m, \tag{9}$$

over the mentioned partitions. Thus from (8) and (9) we deduce the following representation [22]:

$$m!\ a_m = Y_m(-0!\ s_1, -1!\ s_2, -2!\ s_3, -3!\ s_4,\ ...,\ -(m-2)!\ s_{m-1},\ -(m-1)!\ s_m). \tag{10}$$

We know the interesting relation [26-28]:

$$\left[\frac{d^m}{dt^m} e g^{(t)}\right](0) = e^{g(0)}\ Y_m(g^{(1)}(0),\ g^{(2)}(0),\ ...,\ g^{(m)}(0)), \tag{11}$$

then (6), (7) and (11) imply (10) if we employ $g(t) = f_m(t)$

In [23, 29] we find the following expression for the Bell polynomials:

$$Y_m(x_1, x_2,\ ...,\ x_m) = \begin{vmatrix} \binom{m-1}{0}x_1 & \binom{m-1}{1}x_2 & \cdots & \binom{m-1}{m-2}x_{m-1} & \binom{m-1}{m-1}x_m \\ -1 & \binom{m-2}{0}x_1 & \cdots & \binom{m-2}{m-3}x_{m-2} & \binom{m-2}{m-2}x_{m-1} \\ 0 & -1 & \cdots & \binom{m-3}{m-4}x_{m-3} & \binom{m-3}{m-3}x_{m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \binom{1}{0}x_1 & \binom{1}{1}x_2 \\ 0 & 0 & \cdots & -1 & \binom{0}{0}x_1 \end{vmatrix}, \tag{12}$$

therefore:

$$Y_0 = 1,\ Y_1 = x_1,\ Y_2 = x_1^2 + x_2,\ Y_3 = x_1^3 + 3x_1x_2 + x_3,\ \ Y_4 = x_1^4 + 6\,x_1^2\,x_2 + 4\,x_1\,x_3 + 3\,x_2^2 + x_4,$$

$$Y_5 = x_1^5 + 10\,x_1^3\,x_2 + 10\,x_1^2\,x_3 + 15\,x_1\,x_2^2 + 5\,x_1\,x_4 + 10\,x_2\,x_3 + x_5,\ \ ... \tag{13}$$

We see that (13) implies (4) if we employ $x_1 = -s_1$, $x_2 = -s_2$, $x_3 = -2\ s_3$, $x_4 = -6\ s_4$, $x_5 = -24\ s_5$, ..., which reproduces (10).

In fact, it is simple to prove that (12) with gives (5), thus the coefficients of the characteristic equation (1) are generated by the complete Bell polynomials.

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