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Bach, Cotton and Lanczos Tensors in Gödel Geometry

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Abstract: We exhibit that in Gödel spacetime the Lanczos potential is essentially the Cotton tensor and we show its relationship with the Bach tensor.

Key words: Gödel spacetime · Lanczos potential · Bach and Cotton tensors

INTRODUCTION

In [1] was proved that the Gödel's metric [2, 3]:

$$ds^2 = (dx^0)^2 + e^{x^3} dx^0 dx^1 + \frac{1}{2} e^{x^3} (dx^1)^2 - (dx^2)^2 - (dx^3)^2, \quad (1)$$

admits the Lanczos potential [4]:

$$K^{abc} = -\frac{2}{9} C^{abcr};r, \quad (2)$$

where C_{abcd} is the corresponding Weyl tensor.

On the other hand, the Cotton tensor is given by [5]:

$$C^{cab} = 2C^{abcr};r, \quad (3)$$

which plays an important role in the study of three-dimensional spaces [6-8], hence in Gödel geometry the Lanczos generator is proportional to (3):

$$K_{abc} = \frac{1}{9} C_{cab}. \quad (4)$$

The tensor (2) has the following algebraic properties [4]:

$$K_{abc} = K_{bac}, \;\; K_{abc} + K_{bca} + K_{cab} = 0, \;\; K^b_{ab} = 0, \quad (5)$$

and verifies the differential gauge:

$$K^{abc};c = 0, \quad (6)$$

which allows to construct the symmetric tensor:

$$K^{ac} = K^{abc};b, \quad K^a_a = 0. \quad (7)$$

In Sec. 2 we obtain the connection between (7) and the Bach tensor [5, 9, 10]:

$$B^{ac} = C^{abcr};rb - \frac{1}{2} C^{abcr} R_{br}, \quad (8)$$

where $R_{br} = R^d{}_{bdr}$ is the Ricci tensor, with the symmetries:

$$B_{ac} = B_{ca}, \quad B^a_a = 0, \quad B^{ac}_{;c} = 0. \quad (9)$$

In four dimensions, the Bach tensor is the only 2-index tensor (up to constant rescaling) which is symmetric, divergence-free and quadratic in the Riemann curvature tensor or linear combination of its second derivatives. The origin of the Bach tensor is in an integrability condition for a four-dimensional space to be conformal to an Einstein space.

Bach Tensor: From (2), (7) and (8):

$$-\frac{9}{2} K^{ac} = B^{ac} + \frac{1}{2} C^{abcr} R_{br} = B^{ac} + \frac{1}{2} R^{abcr} G_{br} - \frac{5}{12} R^{ac} - \frac{1}{6} g^{ac}, \quad (10)$$

such that $R_{ad} R_c{}^d = -R_{ac}$, $R = R_a{}^a = -1$ and $G_{ac} = R_{ac} + \frac{1}{2} g^{ac}$ are the scalar curvature and the Einstein tensor for the Gödel cosmological model [1].

Besides, we have the relation [1]:

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$$R^{abcr}G_{br} = 3K^{ac} - \frac{1}{2}R^{ac}, \tag{11}$$

then (10) implies:

$$B^{ac} = -6K^{ac} + \frac{2}{3}R^{ac} + \frac{1}{6}g^{ac} = \frac{2}{3}E^{ac} - 6K^{ac}, \quad E^{r}_{r} = 0, \tag{12}$$

where $E_{ac} = R_{ac} - \frac{R}{4}g_{ac}$, which establishes a relationship between the Bach and Lanczos tensors in Gödel spacetime. The expression (12) is equivalent to:

$$B^{ac} = \frac{1}{2}U^{ac} + \frac{1}{6}V^{ac}, \quad U^{ac} = -12K^{ac} + E^{ac}, \quad V^{ac} = E^{ac}, \tag{13}$$

being U_{br} and V_{br} the tensors introduced by Bergman [5].

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