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Moore-Penrose's Inverse and Solutions of Linear Systems

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Abstract: We employ the generalized inverse matrix of Moore-Penrose to study the existence and uniqueness of the solutions for over- and under-determined linear systems, in harmony with the least squares method.

Key words: Linear systems · SVD · Least squares technique · Pseudoinverse of Moore-Penrose

INTRODUCTION

For any real matrix A_{nxm}, Lanczos [1, 2] introduces the matrix:

$$S_{(n+m)x(n+m)} = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}, \qquad (1)$$

with A^T denoting the transpose matrix and studies the eigenvalue problem:

$$S\vec{\omega} = \lambda\vec{\omega}, \qquad (2)$$

where the proper values are real because S is a real symmetric matrix. Besides:

rank $A \equiv p$ = Number of positive eigenvalues of S, (3)

such that $1 \le p \le \min(n, m)$ Then the singular values or canonical multipliers follow the scheme:

$$\lambda_1, \lambda_2, ..., \lambda_p, -\lambda_1, -\lambda_2, ..., -\lambda_p, 0, 0, ..., 0, \qquad (4)$$

that is, $\lambda = 0$ has the multiplicity $n + m - 2p$. Only in the case $p = n = m$ can occur the absence of the null eigenvalue.

The proper vectors of S, named 'essential axes' by Lanczos, can be written in the form:

$$\vec{\omega}_{(n+m)x1} = \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix}\begin{matrix} n \\ m \end{matrix}, \qquad (5)$$

then (1) and (2) imply the Modified Eigenvalue Problem:

$$A_{nxm}\vec{v}_{mx1} = \lambda\vec{u}_{nx1}, \quad A^T_{mxn}\vec{u}_{nx1} = \lambda\vec{v}_{mx1}, \qquad (6)$$

hence:

$$A^T A\vec{v} = \lambda^2\vec{v}, \quad AA^T\vec{u} = \lambda^2\vec{u}, \qquad (7)$$

with special interest in the associated vectors with the positive eigenvalues because they permit to introduce the matrices:

$$U_{nxp} = (\vec{u}_1, \vec{u}_2, ..., \vec{u}_p), \quad V_{mxp} = (\vec{v}_1, \vec{v}_2, ..., \vec{v}_p), \qquad (8)$$

verifying $U^T U = V^T V = I_{pxp}$ because:

$$\vec{u}_j . \vec{u}_k = \vec{v}_j . \vec{v}_k = \delta_{jk}, \qquad (9)$$

therefore $\vec{\omega}_j . \vec{\omega}_k = 2\delta_{jk}, \ j, k = 1, 2, ..., p$. Thus, the Singular Value Decomposition (SVD) express [1-5] that A is the product of three matrices:

$$A_{nxm} = U_{nxp}\, \Lambda_{pxp}\, V^T_{pxm} \quad \Lambda = \text{Diag}\,(\lambda_1, \lambda_2, ... \lambda_p) \qquad (10)$$

This relation tells that in the construction of A we do not need information about the null proper value; the information from $\lambda = 0$ is important to study the existence and uniqueness of the solutions for a linear system associated to A. Golub [6] mentions that the SVD has played a very important role in computations, in solving least squares problems [7], in signal processing problems and so on; it is just a very simple decomposition, yet it is of fundamental importance in many problems arising in technology.

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It is important to observe that the symmetric matrices $(UU^T)_{nxn}$ and $(VV^T)_{nxm}$ are identity matrices for arbitrary vectors into their respective spaces of activation [5], that is:

$$UU^T\vec{u}=\vec{u},\quad \forall\vec{u}\in Col\ U,\quad VV^T\vec{v}=\vec{v},\quad \forall\vec{v}\in Col\ V; \tag{11}$$

besides, (10) allows obtain the SVD of the Gram matrices:

$$(AA^T)_{nxn}=U\Lambda^2U^T,\quad (A^TA)_{mxm}=V\Lambda^2V^T, \tag{12}$$

such that $p=rank\ A=rank\ (AA^T)=rank\ (A^TA)$.

From (10) and (12) we observe that:

$$Col\ A=Col\ (AA^T)=Col\ U,\quad Col\ A^T=Col\ (A^TA)=Col\ V. \tag{13}$$

The eigenvectors associated with λ = 0 verify the equations:

$$A\vec{v}_j=\vec{0},\ j=1,...,m-p,\quad A^T\vec{u}_k=\vec{0},\ k=1,...,n-p,$$
$$\vec{v}_r.\vec{v}_j=0,\ \forall r,j,\ \vec{u}_t.\vec{u}_k=0,\ \forall t,k \tag{14}$$

therefore:

$$V^T\vec{v}_j=\vec{0},\ \forall j,\quad U^T\vec{u}_k=\vec{0},\ \forall\ k, \tag{15}$$

$$A\vec{x}\in Col\ U\ and\ A^TA\vec{x}\in Col\ V,\ \forall\vec{x}\in E^m,$$
$$A^T\vec{y}\in Col\ V\ and\ AA^T\vec{y}\in Col\ U,\ \forall\vec{y}\in E^n.$$

In Sec. 2 we exhibit the Moore-Penrose's pseudoinverse of A [8-13] via the corresponding SVD [14-16], which is useful in Sec. 3 to study the solutions of over- and under-determined linear systems [2, 5] in the spirit of the least squares method [7, 17].

Generalized Inverse: The Moore-Penrose's inverse [2, 8-13] is given by:

$$A^+_{mxn}=V_{mxp}\Lambda^{-1}_{pxp}U^T_{pxn}, \tag{16}$$

which coincides with the natural inverse obtained by Lanczos [2, 5]. The matrix (16) satisfies the relations [10, 11, 13]:

$$AA^+=A,\ A^+AA^+=A^+,\quad (AA^+)^T=AA^+,\ (A^+A)^T=A^+A, \tag{17}$$

that characterize the pseudoinverse of Moore-Penrose. In particular, from (10), (11) and (16):

$$AA^+=UU^T\quad \therefore\quad AA^+\vec{u}=\vec{u},\ \forall\vec{u}\in Col\ U,$$
$$A^+A=VV^T\quad \therefore\quad A^+A\vec{v}=\vec{v},\quad \forall\vec{v}\in Col\ V. \tag{18}$$

The use of (8) and (10) into (16) implies the following expression for the Lanczos generalized inverse:

$$A^+=(\vec{t}_1\vec{t}_2...\vec{t}_n),\ \vec{t}_j=\frac{u_1^{(j)}}{\lambda_1}\vec{v}_1+\frac{u_2^{(j)}}{\lambda_2}\vec{v}_2+...+\frac{u_p^{(j)}}{\lambda_p}\vec{v}_p,$$
$$j=1,...,n, \tag{19}$$

where $u_k^{(j)}$ means the j th- component of $\vec{u}_k$; similarly:

$$(A^+)^T=(\vec{r}_1\vec{r}_2...\vec{r}_n),\ \vec{r}_k=\frac{v_1^{(k)}}{\lambda_1}\vec{u}_1+\frac{v_2^{(k)}}{\lambda_2}\vec{u}_2+...+\frac{v_p^{(k)}}{\lambda_p}\vec{u}_p,$$
$$k=1,...,m, \tag{20}$$

therefore:

$$Col\ A^+=Col\ V,\ Col(A^+)^T\equiv Col(U\Lambda^{-1}V^T)=Col\ U. \tag{21}$$

We can use (16) to construct the pseudoinverse of each Gram matrix, in fact [13]:

$$(A^TA)^+_{mxm}=V\Lambda^{-2}V^T,\ (AA^T)^+_{nxn}=U\Lambda^{-2}U^T, \tag{22}$$

with the interesting properties:

$$(A^TA)^+A^T=A^+,\ (AA^T)^+A=(A^+)^T,$$
$$(A^TA)^+(A^TA)=A^+A=VV^T. \tag{23}$$

Each matrix has a unique inverse because every matrix is complete within its own spaces of activation. The activated p-dimensional subspaces (eigenspaces / operational spaces) are uniquely associated with the given matrix [5].

Linear Systems: We want to find $\vec{x}\in E^m$ verifying the linear system:

$$A\vec{x}=\vec{b}, \tag{24}$$

for the data A_{nxm} and $\vec{b}\in E^n$. It is convenient to consider two situations:

a). *Over-determined linear system* [2, 5]: In this case we have more equations than unknowns, that is, $m < n$.

Lanczos [18] comments that the ingenious method of least squares makes it possible to adjust an arbitrarily over-determined and incompatible set of equations. The problem of minimizing $(A\vec{x} - \vec{b})^2$ has always a definite solution, no matter how compatible or incompatible the given system is. The least square solution of (24) satisfies [5, 17]:

$$A^T A\vec{x} = A^T \vec{b}, \quad \vec{x} \in Col\ V, \quad p = m, \tag{25}$$

and the remarkable fact about (25) is that it always gives an even-determined (balanced) system, no matter how strongly over-determined the original system has been.

The system (25) is compatible because from (13) and (15) we have that $A^T \vec{b}$ is into $Col\ (A^T A) = Col\ V$. Now we multiply (25) by $(A^T A)^+$ and we use (11) and (23) to obtain the solution:

$$\vec{x} = A^+ \vec{b}, \tag{26}$$

which is unique because $p = m$, that is, $Col\ V = E^m$, then in (14) the system $A\vec{v}_j$ only has the trivial solution; hence the Moore-Penrose's inverse gives the least square solution of (24). The expression (26) is in harmony with the results in [19-22].

We have eliminated over-determination (and possibly incompatibility) by the method of multiplying both sides of (24) by A^T. The unique solution thus obtained coincides with the solution generated with the help of A^+ [5].

b). *Under-determined linear system* [2, 5]: There are more unknowns than equations, that is, $n < m$.

In this case we may try the least square formulation of (24), that is, to accept (26), however, now the solution is not unique because $p < m$ and the system $A\vec{v}_j$ has $m - p$ non-trivial independent solutions; an under-determined system remains thus under-determined, even in the least square approach.

An alternative process is to transform the original $\vec{x}$ into the new unknown $\vec{z}$ via the relation [5]:

$$\vec{x} = A^T \vec{z}, \tag{27}$$

then (24) acquires the structure $AA^T \vec{z} = \vec{b}$ whose least square solution is given by the pseudoinverse of Moore-Penrose:

$$\vec{z} = (AA^T)^+ \vec{b} + \sum_{j=1}^{n-p} c_j \vec{z}_j, \tag{28}$$

where the quantities c_j are arbitrary and the $\vec{z}_j$ are $n - p$ independent vectors generating the *Kernel* (AA^T) = *Kernel* (A^T) [13], that is:

$$A^T \vec{z}_j = \vec{0}, \quad j = 1, \ldots, n - p. \tag{29}$$

Thus, from (16), (22), (23), (28) and (29) we have that the solution of (27) is given by:

$$\vec{x} = A^T (AA^T) + \vec{b} = V\Lambda^{-1} U^T \vec{b} = A^+ \vec{b},$$

in agreement with (26).

Although that (26) is not unique for the under-determined case, we can say that it is the 'natural solution' for the linear system (24).

CONCLUSIONS

Our study shows the importance of the SVD [1-6, 14-16] of a matrix and of the corresponding Moore-Penrose's inverse [8-13], to elucidate the least square solution [7, 17-22] for over- and under-determined linear systems [2, 5].

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