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# Moore-Penrose Pseudoinverse via Faddeev-Sominsky Algorithm

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**Abstract:** The Faddeev-Sominsky method is applied to $A_{nxm}\,A^T{}_{mxn}$ to construct the Moore-Penrose inverse of $A$, in according with the result obtained by Piziak; the Singular Value Decomposition of $A$ is important in our approach.

**Key words:** Generalized inverse · Characteristic polynomial · Cayley-Hamilton theorem · Faddeev-Sominsky's method · Moore-Penrose's inverse · Gram matrices

## INTRODUCTION

For an arbitrary matrix $A_{nxm}$, its Moore-Penrose (MP) generalized inverse $A^+$ [1-4] verifies the properties [2, 4-7]:

$$AA^+ A = A, \quad A^+AA^+ = A^+, \quad (AA^+)^T = AA^+, \quad (A^+A)^T = A^+A \tag{1}$$

and is given by [2, 4, 8, 9]:

$$A^+_{mxn} = V_{mxp}\Lambda^{-1}{}_{pxp}U^T_{pxn}, \tag{2}$$

in terms of the matrices $U,\Lambda$ and $V$ generated by the corresponding Singular Value Decomposition (SVD) of $A$ [8, 10-15], such that $p = rank\ A$ and:

$$A = U\,\Lambda\,V^T, \quad U^TU = V^TV = I_{pxp}, \quad \Lambda = Diag\,(\lambda_1,...,\lambda_p),$$
$$\lambda_j > 0, \quad j = 1,...,p. \tag{3}$$

The Faddeev-Sominsky (FS) procedure [16-19] to obtain $E^{-1}_{nxn}$ is a sequence of algebraic computations on the powers $E^r$ and their traces, in fact, this algorithm is given via the instructions:

$$\begin{aligned}
&B_1 = E, && e_1 = -trB_1, && C_1 = B_1 + e_1 I,\\
&B_2 = C_1E, && e_2 = -\frac{1}{2}trB_2, && C_2 = B_2 + e_2 I,\\
&\vdots && \vdots && \vdots\\
&B_{n-1} = C_{n-2}E, && e_{n-1} = -\frac{1}{n-1}trB_{n-1}, && C_{n-1} = B_{n-1} + e_{n-1}I,\\
&B_n = C_{n-1}E, && e_n = -\frac{1}{n}trB_n, &&
\end{aligned} \tag{4}$$

and if $E$ is non-singular, then:

$$E^{-1} = -\frac{1}{e_n}C_{n-1}. \tag{5}$$

because $e_n = (-1)^n \det E$.

From (4) we can see that [20]:

$$C_k = E^k + e_1 E^{k-1} + e_2 E^{k-2} + ... + e_{k-1} E + e_k I, \; k = 1,2,...,n-1,$$
$$C_n = B_n + e_n I = O, \tag{6}$$

and for $k = n - 1$:

$$C_{n-1} = E^{n-1} + e_1E^{n-2} + e_2E^{n-3} + ... + e_{n-2}E + e_{n-1}I = -e_nE^{-1},$$

in harmony with (5) because the property $C_n = O$ is equivalent to the Cayley-Hamilton-Frobenius theorem [21-24]:

$$E^n + e_1E^{n-1} + e_2E^{n-2} + ... + e_{n-1}E + e_nI = O. \tag{7}$$

If $E$ is singular, the process (4) gives the adjoint matrix of $E$ [17], in fact, $Adj\ E = (-1)^{n+1}\ C_{n-1}$.

In Sec. 2 we apply this FS algorithm to the Gram matrix $AA^T$ to deduce the result of Piziak [25] for the MP inverse of $A$.

**MP Pseudoinverse via FS Technique:** Here we employ the FS method for:

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$$E = AA^T \overset{(3)}{=} U\Lambda^2 U^T, \tag{8}$$

and we accept that $k$ is the largest index with $e_k \neq 0$ in (7), thus:

$$E^n + e_1 E^{n-1} + \ldots + e_{k-1} E^{n-k+1} + e_k E^{n-k} = O, \tag{9}$$

therefore:

$$\Lambda^{2n} + e_1 \Lambda^{2(n-1)} + \ldots + e_{k-1} \Lambda^2 (n-k) = O,$$

which can be multiplied by $\Lambda^2(k-1-n)$ to obtain the relation:

$$\Lambda^{2(k-1)} + e_1 \Lambda^{2(k-2)} + \ldots + e_{k-2}\Lambda^2 + e_{k-1} I + e_k \Lambda^{-2} = O. \tag{10}$$

On the other hand, from (6) and (8):

$$C_{k-1} = U[\Lambda^{2(k-1)} + e_1 \Lambda^{2(k-2)} + \ldots + e_{k-2}\Lambda^2]$$
$$U^T + e_{k-1} I \overset{(10)}{=} -U(e_{k-1} I + e_k \Lambda^{-2}) U^T + e_{k-1}, I,$$

hence:

$$A^T C_{k-1} \overset{(3)}{=} V\Lambda U^T C_{k-1} = -e_k V \Lambda^{-1} U^T = -e_k A^+,$$

which is the interesting result of Piziak [25]:

$$A^+ = -\frac{1}{e_k} A^T C_{k-1}, \tag{11}$$

that is, the FS method gives the MP inverse of $A_{nxm}$ if it is applied to the Gram matrix $AA^T$.

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