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# An Identity for Stirling Numbers of the First Kind

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**Abstract:** We show an identity involving Stirling numbers of the first kind.

**Key words:** Stirling numbers

## INTRODUCTION

Here we obtain the following identity for Stirling numbers of the first kind [1-3]:

$$A \equiv \sum_{j=0}^{n-m} \binom{m+j}{j} m^j S_n^{(m+j)} = n! \sum_{r=0}^{m} \frac{\binom{m}{r}}{(n-r)!} S_{n-r}^{(m)}, \quad n \geq m \geq 0. \tag{1}$$

In fact, first we remember the properties:

$$S_n^{(k)} = 0, \;\; k > n, \qquad B(m,r) \equiv \sum_{j=0}^{r} m^r S_r^{(j)} = \frac{m!}{(m-r)!}, \tag{2}$$

in particular, $B(m, r) = 0$ and $r!$ for $0 \leq m < r$ and $m = r$, respectively.

If now we employ the relation [2]:

$$\binom{m+j}{j} S_n^{(m+j)} = \sum_{r=0}^{n} \binom{n}{r} S_r^{(j)} S_{n-r}^{(m)}, \tag{3}$$

we deduce that:

$$A = \sum_{r=0}^{n} \binom{n}{r} S_{n-r}^{(m)} \sum_{j=0}^{n-m} m^j S_r^{(j)} \overset{(2)}{=} \sum_{r=0}^{n} \binom{n}{r} S_{n-r}^{(m)} \sum_{j=0}^{r} m^j S_r^{(j)},$$

$$\overset{(2)}{=} \sum_{r=0}^{m} \binom{n}{r} \frac{m!}{(m-r)!} S_{n-r}^{(m)} = n! \sum_{r=0}^{m} \frac{\binom{m}{r}}{(n-r)!} S_{n-r}^{(m)} = \text{eq. (1), q.e.d.}$$

For example, from (1) with $m = 1$ we obtain the identity:

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$$\sum_{k=2}^{n} k \; S_n^{(k)} = \frac{(-1)^n n!}{n-1}, \quad n \geq 2, \tag{4}$$

where it was used the known property $S_{n-1}^{(1)} = (-1)^n (n-2)!$

We can give an application of (4), in fact, from [2]:

$$\binom{x}{n} = \frac{1}{n!} \sum_{k=0}^{n} S_n^{(k)} x^k, \tag{5}$$

therefore:

$$n! [\frac{d}{dx} \binom{x}{n}]_{x=1} = \sum_{k=1}^{n} k \; S_n^{(k)} = \sum_{k=2}^{n} k \; S_n^{(k)} + S_n^{(1)} \overset{(4)}{=} \frac{(-1)^n n!}{n-1} + (-1)^{n-1} (n-1)!,$$

hence:

$$[\frac{d}{dx} \binom{x}{n}]_{x=1} = \begin{cases} 0, & n = 0, \\ 1, & n = 1, \\ \frac{(-1)^n}{n(n-1)}, & n \geq 2. \end{cases} \tag{6}$$

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