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Generalization of the Famous Riccati and Bernoulli ODEs

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Abstract: Proposing a generalization of the famous Riccati and Bernoulli ordinary differential equations (ODEs) by introducing a class of nonlinear first order ODEs. The author provides the general solutions for these introduced classes of ODEs. Besides, some examples to illustrate the applications are provided.

Key words: First order ODE · Riccati and Bernoulli ODE

INTRODUCTION

An ODE of first order is an algebraic equation of $f\left(x, y, \frac{dy}{dx}\right) = 0$, involving derivatives of some unknown function with respect to one independent variable [1-3]. The first order linear ODE on the unknown y can be expressed in a normal (or explicit) form as

$$\frac{dy}{dx} + p(x)y = f(x) \tag{1.1}$$

where $p(x)$ and $f(x)$ are both continuous functions and $p(x)$ is called the coeficient of linear first order ODE [4, 5]. The analytic solution of (1.1) with integral constant C, can be expressed in the form

$$y(x) = e^{-\int P(x)dx}\left(\int e^{\int P(x)dx} f(x)dx + C\right),\ C \in \mathbb{R}.$$

In 1695 Jacob Bernoulli [6] first proposed the study of nonlinear ODE

$$\frac{dy}{dx} + p(x)y = f(x)y^a, \tag{1.2}$$

in which $a \in \mathbb{R}$ is fixed. If $a = 0$ and $a = 1$, then (1.2) is linear, otherwise it is non-linear. If $a \neq 0, 1$ we set $u = y^{1-a}$, then the equation will be reduced to (1.1), which will be easily solved. Thus,

$$\frac{dy}{dx} + (1-a)p(x)u = (1-a)f(x).$$

The first order ODE of (1.2), was generalized by the recent work done [7] as:

$$\frac{dy}{dx} + p(x)h(y) = f(x)g(y). \tag{1.3}$$

Where $p(x)$ and $f(x)$ are both continuous and also are the functions $h(y)$ and $g(y)$ with $g(y) \neq 0$. If $g(y)$ is a differentiable function and $h(y) = g(y)(\int g(y)^{-1}\, dy)$, then (1.3) has a family of solutions [7], which satisfies

$$\int g(y)^{-1}dy = e^{-\int P(x)dx}\left(\int e^{\int P(x)dx} f(x)dx + C\right),\ C \in \mathbb{R}.$$

The study of the Jacopo Francesco Riccati differential equations goes back to the early days of modern mathematical analysis, since such equations represent one of the simplest types of nonlinear ODEs and consequently Riccati equations play an important role in physics, mathematics and engineering sciences [8-11]. The classical Riccati equation

$$\frac{dy}{dx} + p(x)y + f(x)y^2 = g(x), \tag{1.4}$$

is a first-order ODE with a quadratic non-linearity. Then, (1.4) may be transformed into linear equation by a change of variable which a single particular solution, say $y = y_1(x)$, of (1.4) is known. Setting $y = y_1(x) + \frac{1}{v}$, then $\frac{dy}{dx} = \frac{dy_1}{dx} - \frac{1}{v^2}\frac{dv}{dx}$. After some simplifications in (1.4), we obtain:

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$$\frac{dv}{dx} - [p(x) + 2f(x)y_1(x)]v = (x), \qquad (1.5)$$

which is a linear first order ordinary differential equation. The ODE (1.4), can be also transformed into Bernoulli ODE by a change of variable which a particular solution, say y = $y_1(x)$, of this equation is known. Putting y = $y_1(x)$ + v, then $\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dv}{dx}$. . After simplifications with little algebra in (1.4), we have:

$$\frac{dv}{dx} + [p(x) + 2f(x)y_1(x)]v = -f(x)v^2. \qquad (1.6)$$

Then transform this again into linear equation as (1.2), see the way in (1.5) is faster than in (1.6) to transform (1.4) in to (1.1).

Next, we present the main results of this paper which are generalization of (1.2) and (1.4) by introducing a class of nonlinear forst order ODEs. There, we present a family of solutions to a subclass of the ODEs of the main results which have (1.2) and (1.4) as particular cases.

The Main Results: For a differentiable function $h(y(x))$ and $u_1(x)$ is a single particular solution of (1.4), we have the following theorem.

Theorem: The following subclass of first order ODEs hold true.

$$\frac{d}{dx}(h(y)) + P(x)h(y) + f(x)[h(y)]^2 = g(x), \qquad (2.1)$$

has a family of solutions which satisfies

$$\frac{1}{h(y) - u_1(x)} = e^{-\int \kappa(x)dx}\left(\int e^{\int \kappa(x)dx} f(x)dx + C\right).$$

Where $\kappa(x)$ = $-[P(x) + 2f(x)u_1(x)]$ and C is an integral constant.

$$\frac{d}{dx}(h(y)) + P(x)h(y) = f(x)[h(y)]^a, \qquad (2.2)$$

has a family of general implicit solutions

$$(h(y))^{1-\alpha} = e^{-\int \zeta(x)dx}\left((1-\alpha)\int e^{\int \zeta(x)dx} f(x)dx + C\right)$$

Where $\zeta(x) = (1 - a)P(x)$ and C is an integral constant.

Proof: First, we proof the first identity (2.1). We transform the given ODE of (2.1) into (1.4), by a change of variable. Let u = h(y) then we obtain

$$\frac{du}{dx} + P(x)u + f(x)u^2 = g(x). \qquad (2.3)$$

Since (2.3) is the same as (1.4), so $u_1(x)$ is a particular solution of both (1.4) and (2.3). Putting $u = u_1(x) + \frac{1}{v(x)}$ and using (1.5), we get

$$\frac{dv}{dx} - [P(x) + 2f(x)u_1(x)]v = f(x). \qquad (2.4)$$

Applying (1.1) the analytic solution of (2.4), which is also the solution of (2.1) is

$$\frac{1}{h(y) - u_1(x)} = e^{-\int \kappa(x)dx}\left(\int e^{\int \kappa(x)dx} f(x)dx + C\right).$$

Where $\kappa(x)$ = – $[P(x) + 2f(x)u_1(x)]$ and C is an integral constant.
Hence we complete the proof of (2.1).

Next, we proof the second identity (2.2). We transform the given ODE of (2.2) into (1.2), by a change of variable. Let $m = h(y)$, then we have

$$\frac{dm}{dx} + P(x)m = f(x)m^a. \qquad (2.5)$$

Applying (1.2) and setting $u = m^{1-a}$, we obtain

$$\frac{du}{dx} + (1-a)P(x)u = (1-a)f(x). \qquad (2.6)$$

Using (1.1), (2.5) and (2.6) after simplification, the solution of (2.2) is

$$(h(y))^{1-\alpha} = e^{-\int \zeta(x)dx}\left((1-\alpha)\int e^{\int \zeta(x)dx} f(x)dx + C\right)$$

Where $\zeta(x) = (1 - a)P(x)$ and C is an integral constant.
Hence we complete the proof of (2.2).

Examples: Let us show the usefulness of the theorem via some examples. Using theorems (2.1) and (2.2), solve the following ODEs.

Example 3.1: The Riccati ordinary differential equation

$$\frac{dy}{dx}+2xy-y^2=1+x^2,$$

is a particular case of (2.1). Given $u_1(x)$ is a single particular solution. Then, we have $P(x) = 2x, f(x) = -1, g(x) = 1 + x^2, h(y) = y$ and $k(x) = -[P(x) + 2f(x)\, u_1(x)] = 0$. A family of solution to this ODE obtained as.

$$\frac{1}{y-x}=e^{-\int odx}\left(\int -e^{\int 0dx}dx+c_1\right), for\ c_1\in\mathbb{R}.$$

Hence $y(x)=x+\frac{1}{c-x}$, is the required general solution.

Example 3.2: $\frac{dy}{dx}+2x\tan y-\sec y+\cos y=(1+x^2)\sec y$

The given ODE can be rewritten in the form

$$\cos y\frac{dy}{dx}+2x\sin y-(\sin y)^2=1+x^2$$

Given $u_1(x) = x$ is a particular solution and let u = sin y thus $y(x) = arcsin\left(x+\frac{1}{c-x}\right)$ is the solution.

Example 3.3:

$$\frac{dy}{dx}+2x(1+y^2)arctany-(1+y^2)(arctany)^2=(1+x^2)(1+y^2)$$

The required general solution is $y(x)=\tan\left(x+\frac{1}{c-x}\right)$.

Example 3.4: $\frac{dy}{dx}-y^2=-\frac{2}{x^2}$, provided that $u_1(x)=\frac{1}{x}$ is a solution.

This is the classical Riccati ODE such that $P(x) = 0, f(x) = -1$, $g(x)=-\frac{2}{x^2}$, and $k(x)=\frac{2}{x}$.

Hence $y(x)=\frac{1}{x}+\frac{3x^2}{3C-x^3}$ is the general solution and C is an integral constant.

Example 3.5: $\frac{dy}{dx}-y(\ln y)^2=-\frac{2y}{x^2}$. Let $u = \ln y$ and using example (3.4), the general implicit solution is $\ln y=\frac{1}{x}+\frac{3x^2}{3C-x^3}$.

Example 3.6: Solve the Gompertz population growth model:

$$\frac{dy}{dx}=ry\ln\left(\frac{K}{y}\right).$$

The given model can be expressed in the form:

$$\frac{1}{y}\frac{dy}{dx}-r\ln\left(\frac{K}{y}\right)=0.$$

Let $u = \ln(K/y)$ and applying (2.1) after simplification, we obtain:

$$\ln\left(\frac{K}{y}\right)=Ce^{-rx}\Rightarrow y(x)=\frac{K}{e^{Ce-rx}},$$

is the general analytic solution. Where C is an integral constant, x is time, r is intrinsic growth rate of population, $y(x)$ is the size of population at time x and K is the carrying capacity of the environment, then as time goes to very long the population size approach to the carrying capacity of the environment [12], for positive intrinsic growth rate r, which shows the given model is logistic.

Example 3.7: Suppose for $\alpha \neq 0$ and $\alpha \neq 1$ the Bernoulli ODE

$$\frac{dy}{dx}+P(x)\frac{y}{1-a}=f(x)y^{\alpha},$$

is a particular case of (2.2). Here $k(x) = P(x)$. A solution to this ODE is

$$y^{1-\alpha}=(1-\alpha)e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)dx+Ce^{-\int P(x)dx)},C\in\mathbb{R}$$

$$\frac{y^{1-\alpha}}{1-\alpha}=e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)dx+Ce^{-\int P(x)dx},C\in\mathbb{R}.$$

Example 3.8: $\frac{dy}{dx}-\frac{2}{x}y=xy^{\frac{1}{2}}$. Clearly $\alpha=\frac{1}{2},P(x)=-\frac{2}{x},f(x)=x$ and $\kappa(x)=-\frac{1}{x}$ for the given classical Bernoulli ODE. Thus, we get

$y^{\frac{1}{2}} = \frac{1}{2}x\int dx + Cx = \frac{1}{2}x^2 + Cx,$, for integral constant C.

Hence $y(x) = \left(\frac{1}{2}x^2 + Cx\right)^2$ is the desired result.

Example 3.9: $\frac{dy}{dx} - \frac{2}{x}(1+y^2)\tan^{-1} y = x(1+y^2)\sqrt{\tan^{-1} y}$, for $0 < x < \frac{\pi}{2}$.

Hence $y(x) = \tan\left(\frac{1}{2}x^2 + Cx\right)^2$ is the general result and C is an integral constant.

Thus for any appropriate polynomial function $h(y)$, we have

$$(ny^{n-1} + (n-1)y^{n-2} + ... + 2y + 1)\frac{dy}{dx} - \frac{2}{x}(1 + y + y^2 + ... + y^n) = x\sqrt{1 + y + y^2 + ... + y^n}.$$

So that $1 + y + y^2 + ... + y^n = \left(\frac{1}{2}x^2 + Cx\right)^2$, is the general implicit solution.

In general, $\frac{d}{dx}\left(h(y)\right) - \frac{2}{x}h(y) = x\sqrt{h(y)}$ implies that $h(y) = \left(\frac{1}{2}x^2 + Cx\right)^2$.

Example 3.10: A simple logistic population growth model is given as:

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right).$$

where r is the intrinsic growth rate, K is the carrying capacity of the environment, $y(t)$ is the population size at time t and $\frac{dy}{dt}$ is the rate of change of the given population with respect to time t. Then, we have

$$\frac{dy}{dt} - ry = -\frac{r}{K}y^2.$$

Thus, using (2.2), we obtain

$$y^{-1} = e^{-rt}\left(\frac{e^{rt}}{K} + C\right) \Rightarrow y(t) = \frac{K}{1 + KCe^{-rt}}.$$

Here C is an integral constant. Thus as time goes to very long the population size approach to the carrying capacity of the environment [12, 13] for positive intrinsic growth rate.

Concluding Remarks: In this paper besides having an important history background, it also has interesting applications. In particular, the ideas of this paper may be a base to obtain a generalized version of other first order ODEs which is in the progress. Moreover, the approach adopted in this paper was meant to reach not only researchers but also undergraduate students.

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