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Identities of Gessel, Jha and Raabe for Bernoulli Numbers

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Abstract: We employ the generating function of the Bernoulli polynomials to prove a generalization of Raabe's theorem, and after we use the Jha's identity to show the Gessel's expression involving Bernoulli numbers.

Key words: Bernoulli and Stirling numbers - Raabe's theorem - Bernoulli polynomials

INTRODUCTION

The Bernoulli polynomials $B_n(x)$ are generated via the expression [1-5]:

$$\sum_{r=0}^{\infty} B_r(y)\,\frac{q^r}{r!} = \frac{q\,e^{qy}}{e^q - 1}, \tag{1}$$

Therefore:

$$\sum_{r=0}^{\infty} B_r\,\frac{q^r}{r!} = \frac{q}{e^q - 1}, \tag{2}$$

Involving the Bernoulli numbers $B_n = B_n(0)$ [6-11]. Now we employ (1) with $y = mx + \frac{m}{n}\,j,\ q = nt$ and after we apply $\sum_{j=0}^{n-1}$ to obtain:

$$\sum_{r=0}^{\infty} n^r \sum_{j=0}^{n-1} B_r(mx + \frac{m}{n}\,j)\,\frac{t^r}{r!} = \frac{n\,t\,e^{mnxt}}{e^{nt}-1} \sum_{j=0}^{n-1}\,(e^{mt})^j\,,$$

That is:

$$\sum_{r=0}^{\infty} [\,n^{r-1} \sum_{j=0}^{n-1} B_r(mx + \frac{m}{n}\,j)]\,\frac{t^r}{r!} = \frac{t\,e^{mnxt}\,(e^{mnt}-1)}{(1-e^{mt})\,(1-e^{nt})}, \tag{3}$$

Thus it is immediate the symmetry [12]:

$$n^{r-1} \sum_{j=0}^{n-1} B_r(mx + \frac{m}{n}\,j) = m^{r-1} \sum_{k=0}^{m-1} B_r(nx + \frac{n}{m}\,k)\ ,r \geq 0, m, n \geq 1, \tag{4}$$

which for $n = 1$ gives the Raabe's theorem [2, 4, 13-16]:

$$B_k(mx) = m^{k-1} \sum_{r=0}^{m-1} B_k(x + \frac{r}{m}),\ k \geq 0, m \geq 1. \tag{5}$$

Now we consider the quantities:

$$D(m,n) \equiv (-1)^{m+n} \sum_{t=0}^{n} \sum_{q=0}^{m} A(t,q)\, S_{n+1}^{[t+1]}\, S_{m+1}^{[q+1]}\ ,\ A(t,q) \equiv \frac{(-1)^{t+q}\,(t!\,q!)^2}{(t+q+1)!}, \tag{6}$$

Involving Stirling numbers of the second kind [2, 4, 15], hence:

$$F(M,N) \equiv \sum_{n=0}^{N} \sum_{m=0}^{M} D(m,n)\, S_M^{(m)}\, S_N^{(n)}, \tag{7}$$

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$$= \sum_{t=0}^{n} \sum_{q=0}^{m} A(t,q) \sum_{m=q}^{M} (-1)^m S_{m+1}^{[q+1]} S_M^{(m)} \sum_{n=t}^{N} (-1)^n S_{n+1}^{[t+1]} S_N^{(n)}, \quad (8)$$

with the participation of Stirling numbers of the first kind; it is simple to show the property:

$$\sum_{m=q}^{M} (-1)^m S_{m+1}^{[q+1]} S_M^{(m)} = \frac{(-1)^M (M!)^2}{(q!)^2 (M-q)!}, \quad (9)$$

then from (8) and (9):

$$F(M,N) = \frac{(-1)^{M+N} (M!\,N!)^2}{(M+N+1)!} = A(M,N), \quad (10)$$

where it was applied the relation:

$$\sum_{t=0}^{N} \sum_{q=0}^{M} \frac{(-1)^{t+q}}{(t+q+1)!\,(N-t)!\,(M-q)!} = \frac{1}{(M+N+1)!}. \quad (11)$$

We employ (10) in (7), and after we realize the corresponding inversion to deduce the expression:

$$D(m,n) = \sum_{k=0}^{n} \sum_{r=0}^{m} A(k,r) S_n^{[k]} S_m^{[r]} = B_{m+n}, \quad (12)$$

where it was used the Jha's identity [17, 18].

The next step is the analysis of the following quantities:

$$G(m,n) \equiv (-1)^{m+n} \sum_{k=0}^{n} \sum_{r=0}^{m} \binom{n}{k}\binom{m}{r} B_{k+r} = (-1)^{m+n} \sum_{t=0}^{n} \sum_{q=0}^{m} A(t,q) S_{n+1}^{[t+1]} S_{m+1}^{[q+1]}, \quad (13)$$

where were applied the Jha's formula and the relation [14, 15]:

$$\sum_{j=l}^{m} \binom{m}{j} S_j^{[l]} = S_{m+1}^{[l+1]}. \quad (14)$$

Therefore, (6), (12) and (13) imply that $D(m,n) = G(m,n) = B_{m+n}$, that is:

$$B_{m+n} = (-1)^{m+n} \sum_{r=0}^{m} \binom{m}{r} \sum_{k=0}^{n} \binom{n}{k} B_{k+r}, \quad (15)$$

whose binomial inversion implies the Gessel's identity [12, 19]:

$$(-1)^m \sum_{k=0}^{m} \binom{m}{k} B_{n+k} = (-1)^n \sum_{r=0}^{n} \binom{n}{r} B_{m+r}, \; m,n \geq 0. \quad (16)$$

In relation to (16), it is interesting to indicate the Momiyama's relation [12, 20]:

$$(-1)^m \sum_{k=0}^{m} \binom{m+1}{k} (n+k+1) B_{n+k} + (-1)^n \sum_{r=0}^{n} \binom{n+1}{r} (m+r+1) B_{m+r} = 0, m+n \geq 1. \quad (17)$$

REFERENCES

1. Lanczos, C., 1966. Discourse on Fourier series, Oliver & Boyd, Edinburgh.
2. Srivastava, H.M. and J. Choi, 2012. Zeta and q-zeta functions and associated series and integrals, Elsevier, London.
3. Arakawa, T., T. Ibukiyama and M. Kaneko, 2014. Bernoulli numbers and zeta functions, Springer, Japan.
4. Quaintance, J. and H.W. Gould, 2016. Combinatorial identities for Stirling numbers, World Scientific, Singapore.
5. Pita Ruiz C.V., 2016. Carlitz-type and other Bernoulli identities, J. Integer Sequences 19 (2016) Article 16.1.8.
6. Faulhaber, J., 1631. Academiae algebrae.
7. Seki, T., 1712. Essentials of the art of calculation.
8. Jakob Bernoulli, 1713. Ars conjectandi, Basileae 97.

9. Ramanujan, S., 1911. Some properties of Bernoulli numbers, J. Indian Math. Soc., 3: 219-234.
10. Carlitz, L., 1968. Bernoulli numbers, Fibonacci Quart., 6: 71-85.
11. Gould, H.W., 1972. Explicit formulas for Bernoulli numbers, Amer. Math. Monthly, 79(1): 44-51.
12. He, Y., 2013. Symmetric identities for Carlitz's q-Bernoulli numbers and polynomials, Advances in Difference Equations, 246: 1-10.
13. Raabe, J.L., 1851. Zurückführung einiger summen und bestimmten integrale auf die Jacob Bernoullischen funktion, Crelle J., 42: 348-376.
14. Jordan, C., 1957. Calculus of finite differences, Chelsea Pub., New York, pp: 252.
15. Sándor, J. and B. Crstici, 2004. Handbook of number theory. II, Kluwer Academic, Dordrecht, Netherlands.
16. López-Bonilla, J. and R. López-Vázquez, 2016. Bernoulli polynomials, Comput. Appl. Math. Sci., 1(2): 21-22.
17. Jha, S.K., 2020. An identity involving the Bernoulli numbers and the Stirling numbers of the second kind, http://arxiv.org/pdf/2004.12773 Preprint (2020).
18. Jha, S.K. and J. López-Bonilla, 2020. On a recent formula for Bernoulli numbers involving Stirling numbers, Comput. Appl. Math. Sci., 5(2): 25-27.
19. Gessel, I., 2003. Applications of the classical umbral calculus, Algebra Univers, 49: 397-434.
20. Momiyama, H., 2001. A new recurrence formula for Bernoulli numbers, Fibonacci Quart., 39: 285-288.